BASIC MATHEMATICS

Lecture Notes & Tutorials

UNIVERSITY OF NIZWA

FOUNDATION INSTITUTE

BASIC MATHEMATICS

Lecture notes & tutorials

Prepared By:

The team of Mathematics instructors

Joshy Simon (Course Coordinator)
Ablah Al Hinai
Bijoy Jacob
Jossy Joshy
Muhammed Poongadan
Muhammad Waseem

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PREFACE

This book is supplementary material for the Basic Mathematics course offered at the University of Nizwa and it is not a textbook. It contains some lecture notes and tutorials to enhance the teaching-learning process. The prescribed textbook for the course is **Precalculus – Mathematics for Calculus** by *James Stewart, Lothar Redlin and Saleem Watson.*

For the students:

- You are expected to bring this book along with a notebook and other items to each lecture.
- You may answer the tutorial questions in the notebook.
- Some questions are meant for homework purposes, which are to be solved in the same notebook for future reference and exam preparation.
- You will be asked by the instructors to submit your notebook in order to make sure that the home work is being done.
- Please keep in mind that these lecture notes are not a replacement for your textbook. You may refer to the textbook in the library.
- Final answers to selected questions in the tutorials are provided at the end to encourage self-learning and assessment. You are advised to try the tutorial problems by yourself and check the final answers given.
- You may find the Appendices 1 and 2 at the end of the material. Appendix 1 contains the graphs of some important functions and it is intended for a quick reference. Appendix 2 includes the basic mathematical notations and terminologies in English, to provide the students with a smooth transition from the Arabic medium to the English medium.

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CHAPTER 1 – REAL NUMBERS

Learning Objectives:

- ✓ *Describe the set of real numbers, all its subsets and their relationships.*
- ✓ *Identify and use the arithmetic properties of subsets of integers, rational, irrational and real numbers, including closure properties for the four basic arithmetic operations where applicable.*

1.1 The Real Number System

Numbers play a vital role in the day to day life as their applications and uses cannot be limited to the numerical problem solving. There are different types of numbers such as Natural, Integers, Rational, Irrational, Real numbers etc. In this chapter, we will review these types of numbers that make up the real number system.

The Natural Numbers

The numbers 1, 2, 3, 4, 5, … are called the natural numbers (counting numbers). The group of all natural numbers is denoted by **N.** The numbers 1, 3, 5, 7, … are odd natural numbers whereas the numbers 2, 4, 6, 8, … are the even natural numbers.

Whole numbers

The collection of all natural numbers along with zero is known as the set of whole numbers. They are 0, 1, 2, 3, …. It is denoted by **W.**

Integers

The set of all integers is denoted by **Z.** The set of integers include the natural numbers, the number zero and the negatives of the natural numbers. Thus, they are

…, -3, -2, -1, 0, 1, 2, 3, ….

Rational Numbers

The numbers which can be expressed in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$, are called rational numbers. For eg, $\frac{22}{7}, \frac{-3}{5}, \frac{1}{-4}, \frac{-6}{-11}$, -3, 0, 8 etc. are rational numbers. Any terminating or non-terminating recurring decimal number is also a rational number. i.e, Numbers such as 1.75, 2.333…, 0. 2134134134… are rational numbers. The group of rational numbers is denoted by **Q.**

Operations on rational Numbers

Addition of Rational numbers

The sum of two rational numbers is always a rational number. To add two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$, we use the following rule:

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$$

Subtraction of Rational numbers

The difference of two rational numbers is again a rational number. In order to subtract one rational number from another, we proceed in the same way as that of addition.

$$\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$$

Multiplication of Rational numbers

The product of two rational numbers is another rational number whose numerator is the product of the numerators and the denominator is the product of the denominators.

$$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$$

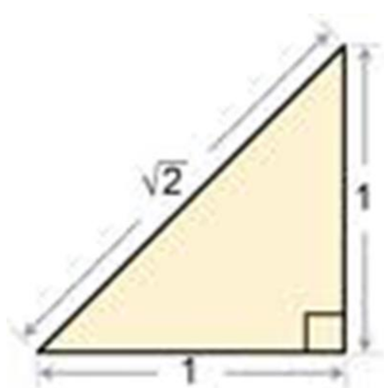
Division of Rational numbers

The ratio of two non-zero rational numbers is again a rational number. To divide a rational number by another rational number, we multiply the first one by the reciprocal of the second.

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$$

Irrational Numbers

Irrational numbers are numbers that cannot be expressed in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. Eg: $\sqrt{2}$, $\frac{1}{\sqrt{3}}$, 4+ π .. etc.

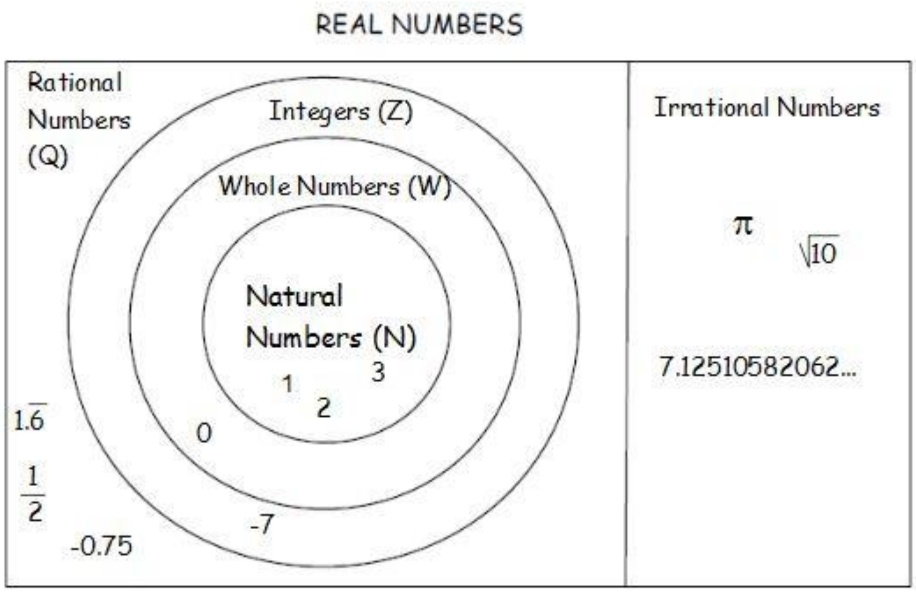


When written in the decimal form, irrational numbers are non-terminating and non-recurring. Thus 4.12432632… is an irrational number.

Note: The value of π is 3.14152965… Since it is approximately equal to $\frac{22}{7}$, we may write $\pi \approx \frac{22}{7}$.

Real Numbers

Real numbers consist of both rational and irrational numbers. The set of all real numbers is denoted by **R.** It is very much evident that the sum, difference, product and quotient of any two non-zero real numbers are real numbers. The following illustrates the family of real numbers.



***Example* 1:** Evaluate: $\frac{3}{7}+\frac{2}{7}$

Solution

$$\frac{3}{7}+\frac{2}{7}=\frac{3+2}{7}=\frac{5}{7}$$

***Example* 2**: Evaluate: $\frac{-3}{5}-\frac{2}{7}$

Solution

$$\frac{-3}{5}-\frac{2}{7}=\frac{-3\times 7-2\times 5}{5\times 7}$$
$$=\frac{-21-10}{35}=\frac{-31}{35}$$

***Example* 3:** Evaluate: $\frac{5}{-2}\times\frac{-3}{10}\times\frac{1}{3}$

Solution

$$\frac{5}{-2}\times\frac{-3}{10}\times\frac{1}{3}=\frac{5\times(-3)\times 1}{(-2)\times 10\times 3}=\frac{-15}{-60}=\frac{1}{4}$$

***Example* 4:** Evaluate: $\frac{3}{8}\div\frac{2}{-5}$

Solution

$$\frac{3}{8}\div\frac{2}{-5}=\frac{3}{8}\times\frac{-5}{2}=\frac{-15}{16}$$

Tutorial – 1

I (1-17) Simplify the following:

1) $\frac{1}{2}+\frac{3}{4}$

2) $\frac{2}{-5}+\frac{5}{6}$

3) $\frac{6}{7}-\frac{4}{5}$

4) $\frac{-8}{15}-\frac{4}{9}$

5) $\frac{5}{12}\times\frac{-3}{10}$

6) $\frac{12}{7}\times\frac{14}{18}$

7) $\frac{20}{-9}\div\frac{1}{18}$

8) $\frac{3}{4}\div 3$

9) $\left(\frac{6}{5}\right)^2-\frac{3}{4}$

10) $\left(\frac{1}{2}-\frac{1}{3}\right)\left(\frac{1}{2}+\frac{1}{3}\right)$

11) $\left(\frac{5}{6}-\frac{3}{8}\right)+\left(1-\frac{2}{3}\right)$

12) $6-\left(\frac{-3}{2}\right)^3$

13) $\left(\frac{1}{3}-\frac{2}{5}\right)\div\left(\frac{1}{3}+\frac{2}{5}\right)$

14) $\left(-\frac{2}{3}-\frac{1}{2}\right)-\left(-\frac{2}{3}\right)^2$

15) $\dfrac{\frac{-1}{12}}{\frac{1}{8}-\frac{1}{9}}$

16) $\dfrac{\frac{4}{7}+\frac{2}{5}}{\frac{4}{7}-\frac{2}{5}}$

17) $\dfrac{1-\frac{1}{3}}{2+\frac{1}{6}}$

II ***Classify the following numbers as rational or irrational:***

a) 7329	*b)* $\sqrt{4}$	*c)* $\sqrt{10}$	*d)* 0.321	*e)* 0.5323232……
f) 8 - $\sqrt{25}$	*g)* 5- $\sqrt{56}$	*h)* $-\frac{4}{5}$	*i)* -3.22	*j)* 0.9583212324…
k) 1.234	*l)* -6	*m)* π	*n)* 3.14	*o)* $\frac{22}{7}$

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1.2 Sets and Intervals

Sets

A set is a well-defined collection of objects, and these objects are called the **elements** of the set. If S is a set, the notation $a \in S$ means that a is an element of S, and $b \notin S$ means that b is not an element of S. For example, if Z represents the set of integers, then $-3 \in Z$ but $\pi \notin Z$.

Representation of sets

1) ***Roster Form (Listing of Elements)***: In this form, the elements of a set are listed in a pair of braces separated by commas.
   ***Example* 1:** The set of even natural numbers less than 10 can be represented as
   A = {2, 4, 6, 8}
   ***Example* 2:** The set of natural numbers can be represented as
   N = {1, 2, 3, 4, …}

2) ***Set-builder Form*:** In this form, instead of listing all the elements, we write the general rule satisfied by the elements of the set.
   ***Example* 3:** The set of real numbers can be written as
   R = {x: x is a real number}

   ***Example* 4:** The set of all vowels of English alphabet can be represented by,
   V = {y: y is a vowel of English alphabet}

Some standard notations for special sets:

Natural numbers	N	=	{1, 2, 3, 4, ...}
Whole numbers	W	=	{0, 1,2, 3, 4, ….}
Integers	Z	=	{…, -3, -2, -1, 0, 1, 2, 3, ….}
Rational numbers	Q	=	$\{x : x = \frac{p}{q}, p, q \in Z, q \neq 0\}$
Real numbers	R		The set of rational and irrational numbers.

The **empty set**, denoted by ϕ or { }, is the set that contains no element.

Operations on Sets

1) **Union of two Sets**:- Let A and B be two sets. Then the union of A and B is the set of all objects that belong to A or B. It is denoted by $A \cup B$, read as 'A union B'.

$$A \cup B = \{x \in U : x \in A \text{ or } x \in B\}$$

***Example* 5:** Let A = {1, 2, 3, 4, 5} and B = {2, 4, 6}
Then, $A \cup B = \{1, 2, 3, 4, 5, 6\}$

2) **Intersection of two sets:-** Let A and B be two sets. Then, the intersection of A and B is the set of all objects that belong to both A and B. It is denoted by $A \cap B$, read as 'A intersection B'.

$$A \cap B = \{x \in U: x \in A \text{ and } x \in B\}$$

***Example 6*:** Let A = {1, 2, 3, 4, 5} and B = {2, 4, 6}
Then, $A \cap B = \{2, 4\}$

***Example* 7:**

If $S = \{1, 2, 3, 4, 5\}$, $T = \{4, 5, 6, 7\}$, and $V = \{6, 7, 8\}$.
Find the sets $S \bigcup T$, $S \bigcap T$ and $S \bigcap V$.

Solution $S \bigcup T = \{1, 2, 3, 4, 5, 6, 7\}$

$S \bigcap T = \{4, 5\}$

$S \bigcap V = \emptyset$.

Intervals

Roughly speaking, an interval is a part of the real numbers that start at one number and stops at another number. For instance, all numbers greater than 1 and smaller than 2 form an interval. Another interval is formed by numbers greater or equal to 1 and smaller than 2 . Thus, when talking about intervals, it is necessary to specify whether the endpoints are part of the interval or not. There are then four types of intervals with three different names: **open, closed and semi-open.** Let us next define them precisely.

(1) The open interval contains neither of the endpoints. If a and b are real numbers, then the open interval of numbers between a and b is written as (a, b).

$$(a, b) = \{x: a < x < b\}$$

(2) The closed interval contains both endpoints. If a and b are real numbers, then the closed interval is written as [a, b].

$$[a, b] = \{x: a \leq x \leq b\}$$

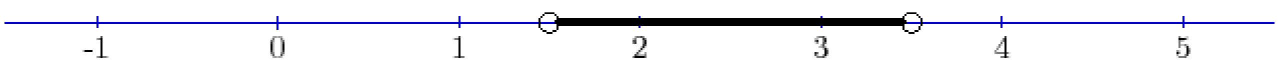
(3) A half-open (or half closed) interval contains only one of the endpoints. If a and b are real numbers, the half-open intervals
(a, b] and [a, b) are defined as

(a, b] contains b and all the real numbers between a and b, whereas
[a, b) contains a and all the real numbers between a and b.

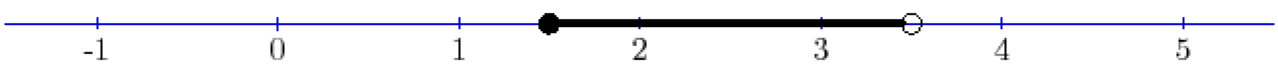
i.e, $(a, b] = \{x: a < x \leq b\}$ and $[a, b) = \{x: a \leq x < b\}$

Note that an interval (a, a) is empty and the interval [a, a] contains only one number a.

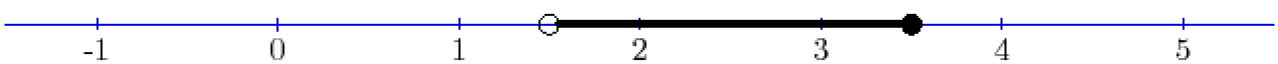
There is a standard way of graphically representing intervals on the real line using filled and empty circles. This is illustrated in the below figures:



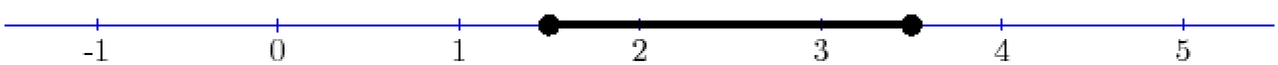
(1.5, 3.5)



[1.5, 3.5)



(1.5, 3.5]



[1.5, 3.5]

The logic is here that an empty circle represents a point not belonging to the interval, while a filled circle represents a point belonging to the interval. For example, the first interval is an open interval.

Infinite intervals

If we allow either (or both) of a and b to be infinite, then we define

(a, ∞) represents all real numbers greater than a

$(-\infty, a)$ represents all real numbers less than a

$[a,\infty)$ represents all real numbers greater than or equal to a and

$(-\infty, a]$ represents all real numbers less than or equal to a

The graphical representation of infinite intervals is as follows:

(1.5, ∞)

[1.5, ∞)

(-∞, 3.5)

(-∞, 3.5]

(-∞, ∞)

Absolute Value of a number

The absolute value of a number is the distance of the number from zero. The modulus of a number a is denoted by | a | and is defined as the numerical part of the number without considering the sign.

Distance between Points on the number line

If a and b are real numbers, then the distance between the points a and b on the real line is

$$d(a,b) = |\ b - a\ |$$

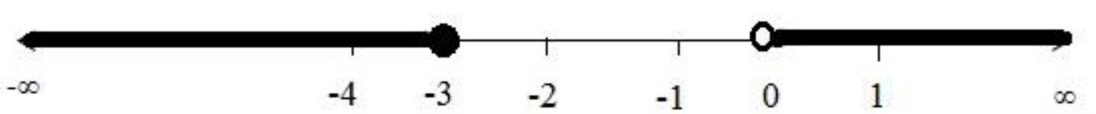
***Example* 8:** Graph the interval.

(a) [-1, 2) (b) (-∞, - 3] ∪ (0, ∞)

Solution

(a) [-1, 2)

(b) (-∞, - 3] ∪ (0, ∞)



***Example* 9:**

(a) $|3| = 3$

(b) $|-3| = -(-3) = 3$

(c) $|0| = 0$

(d) $|3-\pi| = -(3-\pi) = \pi - 3$

***Example* 11:** Find the distance between -7 and 4

Solution

$d(a,b) = |-7 - 4| = |-11| = \mathbf{11}$.

Tutorial – 2

I (1-4) Place the correct symbol (<, >, or =) in the space.

1) $\frac{-5}{2}$ …. 2 2) -6 ….. -10 3) 1.5 ….. $\frac{3}{2}$ 4) |0.34 |…. |-0.34 |

II (5-12) Let $A = \{$ 1, 2, 3, 4, 5, 6 $\}$, $B = \{$ 4, 5, 6, 7 $\}$, and $C = \{$ 6, 7, 8, 9 $\}$.

Perform the indicated operations.

5) $A \bigcup B$ 6) $A \bigcap B$ **7)** $A \bigcap C$ 8) $B \bigcap C$

9) $A \cup B \cup C$ 10) $A \cup (B \cap C)$ 11) $(A \cap B) \cup C$ 12) $A \cap B \cap C$

III (13-17) Express the intervals in terms of inequalities and graph the intervals.

13) $(-\infty, 0)$ 14) $[-3, 0]$ 15) $(-7, 8]$ 16) $[-10, 10]$ 17) $[4, 9)$

IV (18-23) Express the following inequalities as intervals.

18) $\{x \in \mathbb{R}: 2 \le x < 7\}$ 19) $\{x \in \mathbb{R}: -1 < x \le 6\}$ 20) $\{x \in \mathbb{R}: -2 \le x \le 2\}$

21) $\{x \in \mathbb{R}: x < -1.5\}$ 22) $\{x \in \mathbb{R}: x > 3\}$ 23) $\{x \in \mathbb{R}: 0 > x\}$

V (24-29) Evaluate the following:.

24) | - 4 | 25) $-|-7|$ 26) $|-(-6)|$

27) $|12-\pi|$ 28) | -3| - | 3| 29) | - 20| - (- 20) – [(-20) × 3]

VI (30-33) Find the distance between the given numbers.

30) 3 and 20 31) -11 and 17

32) $\frac{13}{2}$ and $-\frac{4}{5}$ 33) – 1.7 and – 2.5

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CHAPTER 2 - EXPONENTS

Learning Objectives:

- ✓ *Demonstrate an understanding of the exponent laws, and apply them to simplify expressions.*
- ✓ *Demonstrate an understanding of the radicals.*

2.1 Exponents

The number $\underbrace{a \times a \times a \times ... \times a}_{b \ \ times}$ can be written as $\boldsymbol{a^b}$ and can be read as *'a raised to b'*. Here we call *a* as the **base** and *b* as the **exponent** (or power). This notation is called **exponential form** or **power notation**.

Similarly, a rational number multiplied several times can be expressed in the same notation.
For eg.

$$\left(\frac{2}{3}\right)^4 = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}$$

The reciprocal of a rational number $\frac{a}{b}$ can be expressed as $\left(\frac{a}{b}\right)^{-1}$ and we know that

$$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$$

Laws of Exponents

1. $a^m \times a^n = a^{m+n}$
2. $a^m \div a^n = a^{m-n}$ or $\frac{a^m}{a^n} = a^{m-n}$
3. $\left(a^m\right)^n = a^{mn}$
4. $(ab)^m = a^m b^m$
5. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
6. $a^{-n} = \frac{1}{a^n}$ In particular, $a^{-1} = \frac{1}{a}$
7. $\left(\frac{a}{b}\right)^{-n} = \frac{a^{-n}}{b^{-n}} = \frac{b^n}{a^n} = \left(\frac{b}{a}\right)^n$
8. For any non-zero real number *a*, $a^0 = 1$

***Example* 1:** Simplify the following:

i) $\left(a^3b^4\right)\left(a^2b\right)$ ii) $\frac{2^9}{2^4}$

iii) $\left(x^2y^3\right)^4$ iv) $\frac{24xyz^3}{-3z^2}$

Solution

i) $\left(a^3b^4\right)\left(a^2b\right) = a^{3+2}b^{4+1} = a^5b^5$

ii) $\frac{2^9}{2^4} = 2^{9-4} = 2^5 = 32$

iii) $(x^2y^3)^4 = (x^2)^4(y^3)^4 = x^{2\times 4}y^{3\times 4} = x^8y^{12}$

iv) $\frac{24xyz^3}{-3z^2} = -8xyz$

***Example* 2:** Show that $\left(a^xb^y\right)\left(\frac{b^{2x}}{a^{-y}}\right) = a^{x+y}b^{y+2x}$

Solution

$$\left(a^xb^y\right)\left(\frac{b^{2x}}{a^{-y}}\right) = a^xb^y.a^yb^{2x}$$

$$= a^xa^yb^yb^{2x}$$

$$= a^{x+y}b^{y+2x}$$

***Example* 3:** Simplify : $\left(4^{-1}+8^{-1}\right)\div\left(\frac{2}{3}\right)^{-1}$

Solution

$$\left(4^{-1}+8^{-1}\right)\div\left(\frac{2}{3}\right)^{-1} = \left(\frac{1}{4}+\frac{1}{8}\right)\div\left(\frac{3}{2}\right)$$

$$= \left(\frac{2}{8}+\frac{1}{8}\right)\div\left(\frac{3}{2}\right)$$

$$= \frac{3}{8}\div\frac{3}{2} = \frac{3}{8}\times\frac{2}{3} = \frac{1}{4}$$

***Example* 4:** Evaluate $\left\{\left(\frac{1}{3}\right)^{-3}-\left(\frac{1}{2}\right)^{-3}\right\}\div\left(\frac{1}{4}\right)^{-3}$

Solution

$$\left\{\left(\frac{1}{3}\right)^{-3}-\left(\frac{1}{2}\right)^{-3}\right\}\div\left(\frac{1}{4}\right)^{-3} = \left\{3^3-2^3\right\}\div 4^3 = \left\{27-8\right\}\div 64 = 19\div 64 = \frac{19}{64}$$

Tutorial – 3

I (1-8) Evaluate each expression.

1) -2^3 2) $(-2)^3$ 3) $(-2)^0$ 4) $\frac{6^5}{6^3}$

5) $\frac{9^{-2}}{3^{-8}}$ 6) $\frac{2^{-2}}{4}$ 7) $\left(\frac{1}{3}\right)^{-3}$ 8) $\left(\frac{1}{2}\right)^4 \times \left(\frac{3}{2}\right)^{-2}$

II (9-34) Simplify the expression and eliminate any negative exponent(s).

9) $a^2 . a^3$

10) $x^4 . x^5$

11) $\frac{y^6}{y^4}$

12) $\frac{b^5}{b^2}$

13) $\frac{p^5}{p^7}$

14) $\frac{q^4}{q^5}$

15) $(x^2)^5$

16) $(a^4)^3$

17) $(a^2b)^3$

18) $(x^3y^2)^4$

19) $\left(\frac{a^4}{b^5}\right)^3$

20) $\left(\frac{x^3}{y^4}\right)^5$

21) $\frac{a^{-2}}{b^{-3}}$

22) $\frac{p^4}{q^{-3}}$

23) $(a^x)(b^{2y})(a^{2x}b^{2y})$

24) $(a^{x-2y+z})(a^{2x-y-z})(a^{x+y+z})$

25) $\frac{-24a^{12}b^9c^5d^2}{2a^6b^3c^4d^2}$

26) $\frac{a^5b^4(a+b)^4}{a^3b^4(a+b)}$

27) $\frac{x^{-6}}{y^9}\frac{y^{-8}}{x^{-12}}$

28) $(ab^2c)^5(-3a^2b)^3$

29) $\left(\frac{a^2b^3}{cd^3}\right)^2 \times \left(\frac{cd^2e}{ab^2}\right)^4$

30) $(2a^2b^3c)\left(\frac{3ab^2}{c^3}\right)^{-2}$

31) $\left(\left(\frac{2}{5}\right)^{-1} - \left(\frac{1}{2}\right)^{-1}\right)^{-2}$

32) $\left(3^{-2} - \left(-\frac{1}{2}\right)^{-1}\right)^{-2}$

33) $\left\{\left[\left(\frac{2}{3}\right)^{-1}\right]^{-1}\right\}^{-2}$

34) $\left\{\left[\left(\frac{-1}{2}\right)^{-2}\right]^{-1}\right\}^{-2}$

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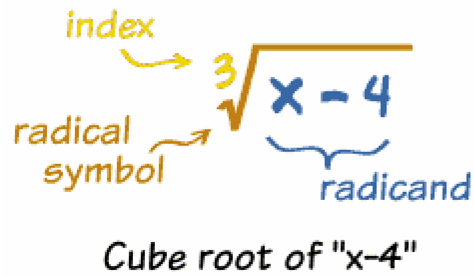
2.2 Radicals

An expression that contains the radical symbol $\sqrt{\ }$ is said to be in the radical form.



The symbol $\sqrt{\ }$ means "the positive root of". "Roots" (or "radicals") are the "opposite" operation of applying exponents; you can "undo" a power with a radical, and a radical can "undo" a power.
For example, $2^2 = 4$, so $\sqrt{4} = 2$.

Similarly, $3^2 = 9$, so $\sqrt{9} = 3$

Thus $\sqrt{a} = b$ means $b^2 = a$.
The expression '$\sqrt{a}$' is read as "root a", "radical a", or "the square root of a".

Definition of nth root

If n is any positive integer, then the principal n^{th} root of a is defined as follows:

$$\sqrt[n]{a} = b \qquad \textbf{means} \qquad b^n = a$$

For Example, $4^2 = 16 \Rightarrow \sqrt[2]{16} = 4$

$4^3 = 64 \Rightarrow \sqrt[3]{64} = 4$

$2^5 = 32 \Rightarrow \sqrt[5]{32} = 2$

In $\sqrt[n]{a}$, 'n' is called as **index** and 'a' is called as **radicand**.

***Note*:** When you simplify the expression '$\sqrt{4}$', the only answer is '+2' even though (-2)× (-2) = 4. Also, $-\sqrt{4} = -2$.

Simplification of radicals:

We say that a square root radical is simplified, or in its simplest form, when the radicand has no square factors.

***Example* 1:** Simplify $\sqrt{33}$.

Solution:

33 has no square factors. Its factors are 3and 11, neither of which is a square number. Therefore, $\sqrt{33}$ is in its simplest form.

***Example* 2:** Simplify $\sqrt{18}$.

Solution: We know that, 18 = 9 × 2.

Therefore, $\sqrt{18}$ is not in its simplest form.

We have, $\sqrt{18} = \sqrt{9\times 2}$

We may now extract, or take out, the square root of 9:

$$\sqrt{18} = \sqrt{9\times 2} = 3\sqrt{2}.$$

***Example* 3:** Simplify $\sqrt{75}$.

Solution:

$$\sqrt{75} = \sqrt{25\times 3} = \sqrt{5\times 5\times 3} = 5\sqrt{3}$$

***Example* 4:** Simplify $\sqrt{180}$.

Solution:

$180 = 18\times 10 = 9 \times 2 \times 10 = 3\times 3 \times 2\times 2\times 5$

Therefore, $\sqrt{180} = \sqrt{3\times 3\times 2\times 2\times 5}$

$$\sqrt{180} = 3\times 2\times \sqrt{5} = 6\sqrt{5}.$$

Properties of n[th] roots		
No.	Property	Example
1	$\sqrt[n]{ab} = \sqrt[n]{a}\ \sqrt[n]{b}$	$\sqrt[3]{8.27} = \sqrt[3]{8}.\ \sqrt[3]{27} = 2\times 3 = 6$
2	$\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\sqrt[4]{\dfrac{16}{81}} = \dfrac{\sqrt[4]{16}}{\sqrt[4]{81}} = \dfrac{2}{3}$
3	$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	$\sqrt{\sqrt[3]{729}} = \sqrt[6]{729} = 3$
4	$\sqrt[n]{a^n} = \begin{cases} a, & if\ n\ is\ odd \\ \lvert a\rvert, & if\ n\ is\ even \end{cases}$	$\sqrt[3]{2^3} = 2$ $\sqrt{4^2} = 4$ $\sqrt{(-4)^2} = 4$

Note: It is very important to notice that

i) $\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$

ii) $\sqrt[n]{a-b} \neq \sqrt[n]{a} - \sqrt[n]{b}$

***Example* 5:** Simplify $\sqrt[3]{x^4}$

$$\sqrt[3]{x^4} = \sqrt[3]{x^3 x} = \sqrt[3]{x^3}\ \sqrt[3]{x} = x\ \sqrt[3]{x}$$

***Example* 6:** $\sqrt{32} + \sqrt{200} = \sqrt{16.2} + \sqrt{100.2}$

$$= \sqrt{16}.\sqrt{2} + \sqrt{100}.\sqrt{2}$$

$$= 4\sqrt{2} + 10\sqrt{2}$$

$$= 14\sqrt{2}$$

Rational Exponent

If the power or the exponent raised on a number is in the form $\frac{m}{n}$, where $n \neq 0$, then the number is said to have rational exponent. To define what is meant by a rational exponent, we need to use radicals.

Let us consider, $\sqrt[n]{a} = b$. This implies $b^n = a$.

Raising both sides to the power $\frac{1}{n}$, we get, $\left(b^n\right)^{\frac{1}{n}} = a^{\frac{1}{n}}$

i.e, $b = a^{\frac{1}{n}}$

Thus, we arrive at the conclusion, $\sqrt[n]{a} = a^{\frac{1}{n}}$

For any rational exponent m/n in lowest terms, where m and n are integers and $n > 0$, we define

$$a^{m/n} = \left(a^{1/n}\right)^m = \left(\sqrt[n]{a}\right)^m \qquad \text{or} \qquad a^{m/n} = \sqrt[n]{a^m}$$

***Example* 7:** Evaluate each of the following.

a) $8^{\frac{2}{3}}$ *b)* $625^{\frac{3}{4}}$

Solution

$$a) \qquad 8^{\frac{2}{3}} = \left(8^2\right)^{\frac{1}{3}}$$
$$= \sqrt[3]{8^2}$$
$$= \sqrt[3]{64}$$
$$= 4$$

$$b) \qquad 625^{\frac{3}{4}} = \left(625^{\frac{1}{4}}\right)^3$$
$$= (5)^3$$
$$= 125$$

Rationalizing the Denominator

It is easy to simplify the radical in a denominator by multiplying both the numerator and denominator by an appropriate expression. This procedure is called **rationalizing the denominator**. If the denominator is of the form $\sqrt{x}$, we multiply the numerator and denominator by $\sqrt{x}$.

***Example* 8:** Rationalize denominators for the following;

a) $\dfrac{3}{\sqrt{2}}$ b) $\dfrac{2}{\sqrt[3]{x^2}}$

Solution

$$a) \qquad \frac{3}{\sqrt{2}} = \frac{3}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{3\sqrt{2}}{2}$$

$$b) \qquad \frac{2}{\sqrt[3]{x^2}} = \frac{2}{(x)^{\frac{2}{3}}} \times \frac{(x)^{\frac{1}{3}}}{(x)^{\frac{1}{3}}}$$
$$= \frac{2x^{1/3}}{x^{\frac{3}{3}}}$$
$$= \frac{2x^{1/3}}{x}$$

Tutorial – 4

I **(1-6) Write each radical expression using exponents, and each exponential expression using radicals.**

	Radical Expression	Exponential Expression
1	$\sqrt[3]{5}$	
2	$\frac{1}{\sqrt{3}}$	
3	$\sqrt[4]{5^3}$	
4		$2^{\frac{5}{2}}$
5		$3^{\frac{-2}{3}}$
6		$\left(\frac{1}{2}\right)^{\frac{-1}{3}}$

II **(7-12) Evaluate each expression.**

7) $\sqrt[5]{32}$

8) $\sqrt[3]{-27}$

9) $\sqrt{\frac{1}{25}}$

10) $\sqrt[3]{\frac{8}{27}}$

11) $\sqrt[3]{\frac{-1}{64}}$

12) $\frac{\sqrt{48}}{\sqrt{3}}$

III **(13-16) Simplify and express as a radical.**

13) $3\sqrt{3}+5\sqrt{2}-2\sqrt{2}+4\sqrt{3}-\sqrt{3}+7\sqrt{2}$

14) $7\sqrt{5}+4\sqrt{2}-4\sqrt{5}+3\sqrt{2}$

15) $\sqrt{32}+\sqrt{18}$

16) $\sqrt{125}-\sqrt{80}$

IV **(17-21) Rationalize the denominator**

17) $\frac{5}{\sqrt{3}}$

18) $\frac{10}{\sqrt{2}}$

19) $\frac{26}{\sqrt{13}}$

20) $\frac{25}{\sqrt{5}}$

21) $\frac{15\sqrt{3}}{\sqrt{5}}$

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CHAPTER 3 - ALGEBRAIC EXPRESSIONS

Learning Objectives:

- ✓ *Perform operations on polynomials and manipulate numerical and polynomial expressions.*
- ✓ *Demonstrate an understanding of algebraic identities and apply them different numerical involving multiplication.*
- ✓ *Perform the term by term division in algebraic expressions.*
- ✓ *Understand factorization of algebraic expressions.*

3.1 Algebraic expressions

In algebra, we come across two types of quantities, namely constants and variables. A symbol having a fixed numerical value is called a *constant* and a symbol which takes various numerical values is known as a *variable*.

Definition of an Algebraic Expression

A combination of constants and variables connected by some or the entire four fundamental operations $+, -, \times$ and $\div$ is called an **algebraic expression**.

Eg: $2x^2 - 3x + 5$, $\sqrt{x} + 6$

Terms

The different parts of the algebraic expression separated by the sign + or – are called the **terms** of the expression. The number present in each term is called the *numerical* **coefficient** *(or* **the coefficient***)*. A term without any variables is called the **constant term**.

Eg: (i) $5 - 3x + 4x^2y$ is an algebraic expression consisting of three terms, namely 5, $-3x$ and $4x^2y$. Here the constant term is 5 and the coefficient of x is -3.

(ii) $7x^2 - 5xy + y^2z - 8$ is an algebraic expression consists of four terms, namely $7x^2$, $-5xy$, y^2z and -8. Here the constant term is -8 and the coefficient of y^2z is 1.

(iii) $3ab$ is an algebraic expression consists of one term, namely $3ab$. Here the coefficient of ab is 3.

Like and Unlike terms of an Algebraic Expression

Two or more terms of an algebraic expression are said to be **like terms** if

(i) they have the same variables and

(ii) the exponents (powers) of each variable are the same.

Otherwise, they are called *unlike terms.*

Eg: (a) $-a$, $7a$ and $5a$ are like terms

(b) The terms $4a^2b^3$, $7b^3a^2$ and $5a^2b^3$ are all like terms

(c) The terms x^3y^2 and x^2y^3 are unlike as the powers of the variables are different.

Polynomial

A **polynomial** is an expression consisting of variables and coefficients, that involves only the operations of addition, subtraction, multiplication, and non-negative integer exponents.

Eg: $3x-4,\ 7x^3+3x-9$

Standard form of a polynomial

The standard form of a polynomial with variable *x* is

$$a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$$

where $a_{0,} a_{1,} ......a_n$ are real numbers and *n* is a non-negative integer.

Degree of a polynomial

The degree of a polynomial is the highest power of the variable in that polynomial.

Eg: (i) The degree of the polynomial $7x^3+3x-9$ is 3.

(ii) The degree of the polynomial $2x^5-3x^4+5x^3-2x^2-9$ is 5

Operations of Algebraic Expressions

The sum of several like terms is another like term whose coefficient is the sum of the coefficients of those like terms.

***Example* 1:**

(i) *Add:* $5x^2 - 7x + 3$, $-8x^2 + 2x - 5$ *and* $7x^2 - x - 2$

(ii) *Simplify:* $(2x^3 - 2x^2 - 2) - (2x^3 - 2x - 1) - (2x^3 - x^2 - x + 1)$

Solution:

(i) $(5x^2 - 7x + 3) + (-8x^2 + 2x - 5) + (7x^2 - x - 2) =$

$$= \underbrace{\left(5x^2 - 8x^2 + 7x^2\right) + \left(-7x + 2x - x\right) + \left(3 - 5 - 2\right)}_{collecting\ \ like\ \ terms}$$

$= (5 - 8 + 7)x^2 + (-7 + 2 - 1)x + \left(3 - 5 - 2\right)$

$= \mathbf{4x^2 - 6x - 4}.$

(ii) $(2x^3 - 2x^2 - 2) - (2x^3 - 2x - 1) - (2x^3 - x^2 - x + 1)$

$= 2x^3 - 2x^2 - 2 - 2x^3 + 2x + 1 - 2x^3 + x^2 + x - 1$

$= (2x^3 - 2x^3 - 2x^3) + (-2x^2 + x^2) + (2x + x) + (-2 + 1 - 1)$

$= \mathbf{- 2x^3 - x^2 + 3x - 2}.$

***Example* 2:**

Multiply: (i) $\left(3x+4\right)\left(2x-1\right)$

(ii) $\left(x-2\right)\left(x^2+3x-1\right)$

Solution:

(i) $\left(3x+4\right)\left(2x-1\right)$

$= 6x^2 - 3x + 8x - 4$

$= 6x^2 + 5x - 4$

(ii) $\left(x-2\right)\left(x^2+3x-1\right)$

$= x^3 + 3x^2 - x - 2x^2 - 6x + 2$

$= x^3 + x^2 - 7x + 2$

Tutorial - 5

I Fill in the following boxes with the no. of terms and the terms in each algebraic expression:

No.	Algebraic Expression	No. of terms	Terms
1	$5a - 3b + 8$		
2	$12a^2b$		
3	$-5x^2y + 2y^2$		

4	$2x - x^2 + 3x^3 - 4x^5 - 7$		
5	7		
6	$x^3 - x$		
7	$2y + 2xy + 2x - 1$		
8	$- m^3n$		

II Identify the coefficients of x, x^2, x^3 and the constant term in each of the following algebraic expressions

No.	Algebraic Expression	Coefficients of			Constant
		x	x^2	x^3	
1	$3x^3 - 2x^2 + 4x - 7$				
2	$\frac{1}{2}x^2 - 3x + 5$				
3	$x^3 - 9$				
4	$x^2 - x^3$				
5	$- x$				
6	$0.5x^3 - 2.5x^2 + 1.5x$				

III Consider the algebraic expression $\mathbf{4a^3 - 3a^2 + a - 8ab}$. Now, answer the following questions:
1) How many terms are there in the above algebraic expression?
2) What is the coefficient of a^2?
3) Which is the constant term in the above expression?

IV Consider the algebraic expression $\mathbf{5x^3 - 2x^2 - 8}$.
Now, answer the following questions:
1) What is the coefficient of x^2?
2) What is the coefficient of x?
3) Which is the constant term in the above expression?

V (1-20) Simplify each of the following algebraic expressions after performing the necessary operations:

1) $(a + b) + (a + b) + (a + b)$

2) $(-8a) + 9a + (-15a) + 4a$

3) $(20\,a - 5b) + (6a + 6b)$

4) $(x^2 - 5) + (x - 3 + x^2)$

5) $(3x^3 - 5x + 4) + (x^2 + 3x - 3x^3)$

6) $(x^2 + xy + y^2 + xy^2) + (-x^2 + 2yx - 3y^2 - x^2y)$

7) $(a - 2b) - (-\,a + 5b)$

8) $(4 - 2x + x^2) - (x^2 - 5x + 3)$

9) $(a^2 - a - 1) - (-a - 3a^2) + (2a^2 - 8)$

10) $2(a - 2b) - 3\,(-\,a + 2b)$

11) $(2x^3 - 2x^2 - 2) - (2x^3 - 2x - 1) - (2x^3 - x^2 - x + 1)$

12) $2(-x + 2y) + 3(x - 2y) - (x + y) + (y - 2x)$

13) $(x - y)x - (y - x)y + xy\,(2 - x - y)$

14) $a(b - c) + b(c - a\,) + c\,(a - b)$

15) $(3t - 2)(5t - 4)$

16) $(4x - 1)(3x + 5)$

17) $(x - 2y)(3x + y)$

18) $(4x - 3y)(2x + 5y)$

19) $(x^3 + 3x^2 - 2)(x^2 + x - 3)$

20) $(x^2y - y^3)(x^3 + xy - y^2)$

****

3.2 Algebraic Identities

An **algebraic identity** is an equality that holds true regardless of the values chosen for its variables. Since identities are true for all valid values of its variables, one side of the equality can be swapped for the other.

Square of a Sum / Difference

If a and b represent some algebraic expressions, then

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$

Caution: Keep in mind that $(a + b)^2 \neq a^2 + b^2$

Difference of two Squares

If a and b represent some algebraic expressions, then

$$a^2 - b^2 = (a + b)(a - b)$$

***Example* 1:**

Evaluate: (i) $(2x + 3)^2$
(ii) $(3m - 2n)^2$

Solution:

(i) $(2x + 3)^2 = (2x)^2 + 2(2x)(3) + 3^2$
$= 4x^2 + 12x + 9$

(ii) $(3m - 2n)^2 = (3m)^2 - 2(3m)(2n) + (2n)^2$
$= 9m^2 - 12mn + 4n^2$

***Example* 2:** **Find the product**

(i) $(4x - 5y)(4x + 5y)$

(ii) $(xy - 5y)(xy + 5y)$

(iii) $(x + 1)^2 - (x - 1)^2$

(iv) $(a - 1)(a + 1)(a^2 + 1)(a^4 + 1)(a^8 + 1)$

Solution:

(i) $(4x-5y)(4x+5y)=16x^2-25y^2$

(ii) $(xy-5y)(xy+5y)=x^2y^2-25y^2$

(iii) $(x+1)^2-(x-1)^2=[(x+1)+(x-1)][(x+1)-(x-1)]$

$$=(2x)(2)$$
$$=4x$$

(iv) $(a-1)(a+1)(a^2+1)(a^4+1)(a^8+1)=(a^2-1)(a^2+1)(a^4+1)(a^8+1)$

$$=(a^4-1)(a^4+1)(a^8+1)$$
$$=(a^8-1)(a^8+1)$$
$$=(a^{16}-1)$$

***Example* 3:**

If $3a-b=5$ and $ab=6$ find $9a^2+b^2$

Solution:

Given that $3a-b=5$ and $ab=6$

$\Rightarrow\quad (3a-b)^2=9a^2-6ab+b^2$

$\Rightarrow\quad 5^2=9a^2-6\times 6+b^2$

$\Rightarrow\quad 25=9a^2-36+b^2$

$\Rightarrow\quad 9a^2+b^2=25+36$

$\Rightarrow\quad 9a^2+b^2=61$

Tutorial - 6

I (1-18) Expand and simplify the following:

1) $(2+x)^2$

2) $(2a+3)^2$

3) $(3-y)^2$

4) $(5-2x^2)^2$

5) $(1+2y)^2$

6) $(3x-4)^2$

7) $(2x^2+3y^2)^2$

8) $\left(c-\frac{1}{c}\right)^2$

9) $(ax + by)^2$

10) $(2ab^2 - 3c^3d^4)^2$

11) $(x + 2y)^2 + 4xy$

12) $(x + 2y)^2 - 4xy$

13) $(a + 2)^2 + (a - 2)^2$

14) $(a + 2)^2 - (a - 2)^2$

15) $(x + 2y)(x - 2y)$

16) $(2x - 3y)(2x + 3y)$

17) $(x^2+a^2)(x^2-a^2)$

18) $(x - y)(x + y)(x^2 + y^2)$

II (19-26) ***Answer the following questions:***

19) If $x+y=4$ and $xy = 5$, find x^2+y^2

20) If $a - b = 3$ and $ab = 5$ find $a^2 + b^2$.

21) Given that $2x + 3y = 10$ and $xy = 4$, find $4x^2 + 9y^2$.

22) If $x-2y=3$ and $xy=5$, find x^2+4y^2

23) If $x-\frac{1}{x} = 3$, find $x^2+\frac{1}{x^2}$.

24) If $x+\frac{1}{x}=2$, find $x^2+\frac{1}{x^2}$

25) If the difference of two numbers is 3 and their product is 4, find the sum of their squares.

26) Find the sum of the squares of two numbers if their sum is 5 and the product is 6.

****

3.3 Factorization of Expressions

Term by Term Division

If a, b and c represent some algebraic expressions, then

$$\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$$

i.e, if a polynomial is divided by a monomial, then each term of the polynomial must be divided by the monomial.

Note: The expression $\frac{a+b}{c}$ is different from $\frac{ab}{c}$. While simplifying the expression $\frac{ab}{c}$,one may evaluate the product ab and then divide this by c or divide a by c and then multiply this by b etc.

***Example* 1:**

(i) Divide: $\frac{2a^3b-6a^2b^2}{2a^2b^2}$

(ii) Divide: $\frac{\left(2a^3b\right)\left(-6a^2b^2\right)}{2a^2b^2}$

Solution

(i) $\frac{2a^3b-6a^2b^2}{2a^2b^2} = \frac{2a^3b}{2a^2b^2}-\frac{6a^2b^2}{2a^2b^2} = \frac{a}{b}-3$

(ii) $\frac{\left(2a^3b\right)\left(-6a^2b^2\right)}{2a^2b^2} = \frac{-12a^5b^3}{2a^2b^2} = -6a^3b$

Factoring a Monomial from a polynomial

Consider an expression of the form *ab + ac*. Clearly, a is a common factor for the terms *ab* and *ac*. Thus we may write:

ab + ac = a(b + c)

***Example* 2:** Factorize the expression $3x^2y^3z – 12x^3y^2$

Solution:
[Here consider the coefficients: 3 and -12. We know that 3 is the greatest common factor for these numbers. So we may factor 3 from them. From x^2 and x^3, we can factor x^2 from them. y^2 can be factored from y^3 and y^2. Since the variable z is not present in the second term, we cannot factor z.]

Thus, $3x^2y^3z – 12x^3y^2 = 3x^2y^2(yz – 4x)$

Remarks:

1. You may verify the result by multiplying.
   i.e, $3x^2y^2(yz – 4x) = 3x^2y^2. yz - 3x^2y^2 . 4x = 3x^2y^3z – 12x^3y^2$

2. You may also factorize the above expression as $3xy(xy^2z – 4x^2y)$. But the expression $xy^2z – 4x^2y$ can be factorized again. So we try to factor the greatest common factor from the terms.

Factoring by Grouping

To factorize an expression with no common factor to all terms and with an even number of terms, we may use the method of grouping as explained below:

Consider the expression, $ac + ad + bc + bd$.

Here, there are no common factors to all the four terms. But, if we group the terms as ($ac + ad$) and ($bc + bd$), we can factor the terms a and b respectively from these groups.

Thus,

$ac + ad + bc + bd$	$= (ac + ad) + (bc + bd)$	[*grouping*]
	$= a(c + d) + b(c + d)$	[*factoring each group*]
	$= (c + d)(a + b)$	[*factoring* (c + d)*from both terms*]

***Example* 3: Factorize the following expressions:**

(i) $5m^2n - 10mn^2 + mn$ (ii) $(2x+4)(x-3)-5(x-3)$

(iii) $3m^2 - 4mn + 6m - 8n$ (iv) $2x^2 + 3x - 2x - 3$

Solution

(i) $5m^2n - 10mn^2 + mn = mn(5m - 10n + 1)$

(ii) $(2x+4)(x-3)-5(x-3) = [(2x+4)-5](x-3) = (2x-1)(x-3)$

(iii) $3m^2 - 4mn + 6m - 8n = (3m^2 - 4mn) + (6m - 8n)$

$= m(3m - 4n) + 2(3m - 4n) = (3m - 4n)(m + 2)$

(iv) $2x^2 + 3x - 2x - 3 = (2x^2 + 3x) - (2x + 3)$ [*notice the sign of the last term*]

$= x(2x + 3) - (2x + 3) = (2x + 3)(x - 1)$

Factoring expressions of the form x^2+px+q

Consider the expressions of the form $x^2 + px + q$. Suppose that

$$x^2 + px + q = (x+a)(x+b)$$

Then, upon multiplying,

$$x^2 + px + q = (x+a)(x+b) = x^2 + (a+b)x + ab$$

Thus $q = ab$ and $p = a + b$, that is, the constant term is the product of two numbers, and the coefficient of x is the sum of these two numbers.

***Example* 4: Factorize the following expressions:**

(i) x^2+5x+6 (ii) x^2-5x+6

(iii) x^2-x-6 (iv) $x^2+10x+24$

(v) $x^2+25x+24$ (vi) $x^2-11x+24$

(vii) $x^2-25x+24$ (viii) $x^2+2x-24$

Solution

(i) $x^2+5x+6=(x+2)(x+3)$

(ii) $x^2-5x+6=(x-2)(x-3)$

(iii) $x^2-x-6=(x-3)(x+2)$

(iv) $x^2 + 10x + 24 = (x+4)(x+6)$

(v) $x^2 + 25x + 24 = (x+1)(x+24)$

(vi) $x^2 - 11x + 24 = (x-3)(x-8)$

(vii) $x^2 - 25x + 24 = (x-1)(x-24)$

(viii) $x^2 + 2x - 24 = (x-4)(x+6)$

Special Factorization

In this section, we discuss the factorization of expressions of the form $a^2 + 2ab + b^2$, $a^2 - 2ab + b^2$ and $a^2 - b^2$.

We use the following identities to factor these expressions:

a) $\boldsymbol{a^2 + 2ab + b^2 = (a+b)^2}$	[Square of a sum]
b) $\boldsymbol{a^2 - 2ab + b^2 = (a-b)^2}$	[Square of a difference]
c) $\boldsymbol{a^2 - b^2 = (a+b)(a-b)}$	[Difference of two squares]

***Example* 5: Factorize the following expressions:**

(i) $9x^2 + 12x + 4$ (ii) $4m^2 - 20mn + 25n^2$

(iii) $16x^2 - 9y^2$ (iv) $x^4 - 9y^2$

(v) $x^3 - 4x$ (vi) $x^4 - 81$

Solution

(i) $9x^2 + 12x + 4 = (3x+2)^2$

(ii) $4m^2 - 20mn + 25n^2 = (2m-5n)^2$

(iii) $16x^2 - 9y^2 = (4x+3y)(4x-3y)$

(iv) $x^4 - 9y^2 = (x^2 - 3y)(x^2 + 3y)$

(v) $x^3 - 4x = x(x-2)(x+2)$

(vi) $x^4 - 81 = (x^2 - 9)(x^2 + 9) = (x-3)(x+3)(x^2+9)$

Tutorial - 7

I (1-8) Simplify the following:

1) $\frac{3x^2 + 5x}{x}$

2) $\frac{(3x^2)(5x)}{x}$

3) $\frac{4x^4 - 6x^2}{2x^2}$

4) $\frac{(4x^4)(-6x^2)}{2x^2}$

5) $\frac{12x^2y^3 + 6x^4y^2}{3x^3y^3}$

6) $\frac{q^2 - pq - pqr}{-q}$

7) $\frac{x^2 - xy - xz}{-x}$

8) $\frac{(x^2)(-xy)(-xz)}{-x}$

II (9-42) Factorize the following:

9) $x^3 - x^2$

10) $3x^2 - 6x$

11) $-2x^3 + 16x$

12) $x^3 - x^2 + x$

13) $6x^2y^3 - 10xy^2$

14) $2x^4 + 4x^3 - 14x^2$

15) $4x^3y - 6xy^3 + 8x^2y^2$

16) $x^4y^3 + x^3y^3 - 5x^2y^3$

17) $x^2 + ax + bx + ab$

18) $ax^2 + a + bx^2 + b$

19) $x^3 - x^2 + x - 1$

20) $x^3 - 2x^2 - 3x + 6$

21) $a^2 - ab - ac + bc$

22) $a^2 - ab + ac - bc$

23) $x^2 + 4x + 3$

24) $x^2 + 7x + 12$

25) $x^2 - 5x + 6$

26) $x^2 - 8x + 12$

27) $x^2 - 8x + 15$

28) $p^2 + 9p - 36$

29) $x^2 - 2x - 15$

30) $r^2 - 21r - 22$

31) $x^3 - 3x^2 - 40x$

32) $x^4 - x^3 - 2x^2$

33) $x^2 + 20x + 100$

34) $x^2 + 12x + 36$

35) $9x^2 + 24xy + 16y^2$

36) $4x^2 - 12x + 9$

37) $m^2 - 4mn + 4n^2$

38) $49a^2 - 16b^2$

39) $16x^2 - 25y^2$

40) $81 - 49y^2$

41) $x^4 - 81y^4$

42) $x^4 - 9y^2$

****

CHAPTER 4 - RATIONAL EXPRESSIONS

Learning Objective:

- ✓ *Simplify rational expressions and rationalize numerators or denominators.*

A rational expression is an expression that can be written in the form $\frac{P}{Q}$ where P and Q are polynomials and Q $\neq 0$.

$$\frac{2}{3},\quad \frac{3y^3}{8},\quad \frac{-4p}{p^3+2p+1},\quad \frac{5x^2-3x+2}{3x+7}$$

are different examples of rational expressions.

The **reciprocal** of a rational expression $\frac{P}{Q}$ is $\frac{Q}{P}$.

Rules for Simplifying Rational Expressions

1. A rational expression is said to be in the simplest form if its numerator and denominator do not have any common factors.
2. We may add, subtract, multiply and divide the rational numbers using the following rules:

- $\frac{P}{Q}+\frac{R}{S}=\frac{PS+QR}{QS}$
- $\frac{P}{Q}-\frac{R}{S}=\frac{PS-QR}{QS}$
- $\frac{P}{Q}\times\frac{R}{S}=\frac{PR}{QS}$
- $\frac{P}{Q}\div\frac{R}{S}=\frac{P}{Q}\times\frac{S}{R}=\frac{PS}{QR}$

***Example* 1:**

Express the rational expression $\frac{x^2-1}{x^2-2x+1}$ in the lowest form.

Solution

$$\frac{x^2-1}{x^2-2x+1}=\frac{(x-1)(x+1)}{(x-1)(x-1)}=\frac{(x+1)}{(x-1)}$$

***Example* 2:**

Express the rational expression $\frac{x^2+5x+6}{x^2-5x-14}$ in the lowest form.

Solution

$$\frac{x^2+5x+6}{x^2-5x-14}=\frac{(x+2)(x+3)}{(x+2)(x-7)}=\frac{(x+3)}{(x-7)}$$

***Example* 3:** Add: $\frac{3}{x-3}+\frac{2}{x+2}$

Solution

$$\frac{3}{x-3}+\frac{2}{x+2}=\frac{3(x+2)}{(x-3)(x+2)}+\frac{2(x-3)}{(x+2)(x-3)}$$

$$=\frac{3x+6}{(x-3)(x+2)}+\frac{2x-6}{(x+2)(x-3)}$$

$$=\frac{5x}{(x-3)(x+2)} \qquad =\frac{5x}{x^2-x-6}$$

***Example* 4:** Multiply: $\frac{x-1}{x+1}\cdot\frac{x+2}{x-2}$

Solution

$$\frac{x-1}{x+1}\cdot\frac{x+2}{x-2}=\frac{(x-1)(x+2)}{(x+1)(x-2)}$$

$$=\frac{x^2+x-2}{x^2-x-2}$$

***Example* 5:** Divide: $\frac{x-1}{x+1}\div\frac{x+2}{x-2}$

Answer:

$$\frac{x-1}{x+1}\div\frac{x+2}{x-2}=\frac{x-1}{x+1}\cdot\frac{x-2}{x+2}$$

$$= \frac{(x-1)(x-2)}{(x+1)(x+2)}$$

$$= \frac{x^2-3x+2}{x^2+3x+2}$$

Tutorial - 8

I (1-20) Simplify the following:

1) $\dfrac{x^2-1}{x^2+2x+1}$

2) $\dfrac{3(x+2)(x-1)}{6(x-1)^2}$

3) $\dfrac{x^2-5x+6}{x^2+5x-14}$

4) $\dfrac{x^2+6x+8}{x^2+5x+4}$

5) $\dfrac{4(x^2-1)}{12(x+2)(x-1)}$

6) $\dfrac{a^2+a}{a^2-1}$

7) $\dfrac{x}{x-2}+\dfrac{2}{x+2}$

8) $2+\dfrac{x}{x+1}$

9) $\dfrac{3}{x-3}+\dfrac{2}{x+2}$

10) $\dfrac{x}{x-1}+\dfrac{x}{x+1}$

11) $\dfrac{1}{x-1}-\dfrac{1}{x+1}$

12) $\dfrac{2}{x+2}-\dfrac{1}{x-1}$

13) $\dfrac{x}{x-4}-\dfrac{3}{x+2}$

14) $\dfrac{4x}{x^2-4}\cdot\dfrac{x+2}{20x}$

15) $\dfrac{x^2-x-12}{x^2-9}\cdot\dfrac{x+3}{x-4}$

16) $\dfrac{a-3}{a^2+6a+9}\cdot\dfrac{a+3}{a^2-9}$

17) $\dfrac{x^2-16}{x^2-9}\cdot\dfrac{x+3}{x-4}$

18) $\dfrac{x^3}{x+1}\div\dfrac{x}{x^2+2x+1}$

19) $\dfrac{x^2+6x+5}{x-1}\div\dfrac{x+5}{x^2-1}$

20) $\dfrac{x^2-4}{x+1}\div\dfrac{x+2}{x-2}$

****

CHAPTER 5 – EQUATIONS

Learning Objectives:

- ✓ *To understand equation, solution of an equation, linear equation in one variable and quadratic equation.*
- ✓ *Translate worded problems into mathematical expression and model simple real life problems with equations.*

An **equation** is a statement that says that two mathematical expressions have the same value.

If all the terms in an equation are numbers, then the equation is said to be an ***arithmetic equation.***

Eg: $3 + 4 = 7$
$5 \times 6 = 30$

If the equation contains a letter **(variable)** that represents an unknown number, then there will be some values of the unknown that make the equation true. Such values are called ***solution (Root) of*** the equation and the process of finding the solutions is called ***solving the equation***.

Eg: If $x - 3 = 2$, then $x = 5$ is the solution (root) of the equation $x - 3 = 2$

5. 1 Linear Equations in one variable

An equation involving only one variable with the highest power of the variable **1** is called a ***linear equation in one variable*** **.**

The general form of a linear equation in one variable is $\boldsymbol{ax+b = 0}$,Where $\boldsymbol{x}$ is the variable and $\boldsymbol{a}$ and $\boldsymbol{b}$ are constants

$$Eg: 6x - 3 = 4 \;,\; 7y - 4 = -3y \;,\; 4t = -9$$

Solution of a linear equation: The value of the variable that makes the equation true is called its ***solution*** or ***root***. A linear equation in one variable has exactly one solution.

Given below are some examples of linear and non-linear equations.

Linear equations	**Nonlinear equations**
$6x - 5 = 9$	$x^2 + 3x = 9$
$y - 7 = 3y$	$\sqrt{x} + 7x = 9$
$\frac{1}{3}x - 9x = 8$	$\frac{3}{x} - 4x = 2$

1) Solving equations by collecting terms

When solving an equation , Whatever you do to an equation do the same thing to both sides of that equation

Suppose we wish to solve the equation for x : $4x + 3 = x + 18$

(i) We can subtract x from each side, because this will remove it entirely from the right, to give

$$4x + 3 - x = x + 18 - x$$
$$3x + 3 = 18$$

(ii) We can subtract 3 from each side to give

$$3x + 3 - 3 = 18 - 3$$
$$3x = 15$$

(iii) We can divide each side by 3

we obtain $x = 5$

So the solution of the equation is $x = 5$. This solution should be checked by substitution into the original equation in order to check that both sides are the same. If we do this, the left is 4(5) + 3 = 23and the right is 5+18 = 23. So the left equals the right and we have checked that the solution is correct.

2) Solving equations by removing brackets & collecting terms

Let us try to solve the equation

$$4(x - 5) - (7 - 2x) = 3(x + 2) - 2(4 - x)$$

We begin by multiplying out the brackets, taking care, in particular, with any minus signs.

$$4x - 20 - 7 + 2x = 3x + 6 - 8 + 2x$$

Write all the x terms in L.H.S and all the numbers in R.H.S

$$4x + 2x - 3x - 2x = 6 - 8 + 20 + 7$$
$$x = 25$$

And again you should take the solution $x = 25$, substitute it back into the original equation to check that we have got the correct answer.

On the L.H.S :

$$4(25 - 5) - (7 - 2 \times 25) = 4 \times 20 + 43 = 123$$

On the R.H.S:

$$3(25 + 2) - 2(4 - 25) = 3 \times 27 + 42 = 123$$

So both sides equal 123. The equation balances and so $x = 25$ is the solution

***Example* 1 : Solve the following linear equations for *x*:**

(i) $x - a + b = 2a + 3b$

(ii) $\frac{x}{2} - \frac{x}{3} = \frac{x}{4} + \frac{1}{2}$

Solution: (i) We have,

$$x - a + b = 2a + 3b$$

$\Rightarrow$ $x = 2a + 3b + a - b$

Collecting the term of x on one side and other terms to the other side

$\Rightarrow$ $x = 3a + 2b$

$\therefore$ The required solution is $\boldsymbol{x = 3a + 2b}$.

(ii) Consider,

$$\frac{x}{2} - \frac{x}{3} = \frac{x}{4} + \frac{1}{2}$$
$$\frac{3x - 2x}{6} = \frac{2x + 4}{8}$$
$$\frac{x}{6} = \frac{x + 2}{4}$$
$$4x = 6x + 12$$
$$4x - 6x = 12$$
$$-2x = 12$$
$$x = -6$$

$\therefore$ The required solution is $\boldsymbol{x = -6}$.

***Example* 2:**

The total resistance R obtained when two resistors R_1 and R_2 are connected parallelly in a circuit is given by the formula $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$. Solve this equation for R_2.

Solution: Consider,

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$$

$$\frac{1}{R}=\frac{R_2+R_1}{R_1R_2}$$

$$R_1R_2=R\left(R_2+R_1\right)$$

$$R_1R_2=RR_2+RR_1$$

$$R_1R_2-RR_2=RR_1$$

$$R_2\left(R_1-R\right)=RR_1$$

$$R_2=\frac{RR_1}{R_1-R}$$

Tutorial - 9

I (1-13) Solve the following linear equations for *x*:

1) $x - 7 = -3$

2) $2x + 7 = 31$

3) $x + 5y = -6y$

4) $x - 3a + b = a + 6b$

5) $4x = -48$

6) $2a\,x = 20a$

7) $\frac{x}{5} = 12$

8) $\frac{x}{a}=a^4$

9) $\frac{ax}{b}=\frac{b}{a}$

10) $\frac{ax}{b}=a^2b^3$

11) $7x = 0$

12) $\frac{1}{2}x - 8 = 1$

13) $a\,x - 4a = 2ab - a$

II (14 - 15) Determine, by substituting, whether the given value is a solution of the equation.

14) $4x + 7 = 9x - 3$; (a) $x = 2$ (b) $x = -2$

15) $1-[2-(3-x)] = 4x - (6+x)$ (a) $x = 2$ (b) $x = 4$

III (16 - 25) Solve the following linear equations for *y*:

16) $6 + y = -10$

17) $y - 4 = -4$

18) $4y + 4 = -44$

19) $6y - 18 = 3$

20) $\frac{16}{y} = 2$

21) $\frac{-18}{2y} = 3$

22) $\frac{2y}{3} + 4 = -10$

23) $\frac{2y}{4} = b + 1$

24) $5y - 3 = 22$

25) $\frac{ay}{b+1} - a^3 = ab$

IV (26 – 28) Answer the following questions:

26) The ideal gas law is given by the equation $PV = nRT$, where P = Pressure, V= Volume, n = number of moles, R = gas constant and T = Temperature. Solve this equation for *R*.

27) Newton's Universal Law of Gravity is given by $F = \frac{GmM}{r^2}$, where *F* is the gravitational force, *m* and *M* are the masses of two objects, *r* is the distance between the centers of the two masses and *G* is the gravitational constant. Solve this equation for 'G'.

28) Solve the following three Equations of Motions for *a*:

i) $v = u + at$ *ii*) $s = ut + \frac{1}{2}at^2$ *iii*) $v^2 = u^2 + 2as$

where, $u =$ initial velocity, $v =$ final velocity, $a =$ acceleration, $t =$ time interval and $s =$ displacement

IV (29 - 35) Solve the following equations for *x*:

29) $2(6x - 4) - 2(2 - 4x) = 1$

30) $\frac{4x-1}{3} = \frac{1-2x}{2}$

31) $\frac{1}{x+2} = \frac{3}{x}$

32) $\frac{2x-1}{x+2} = \frac{4}{5}$

33) $\frac{x+3}{2} = 2x + 1$

34) $\frac{x}{4} - \frac{x}{2} = \frac{x}{3} + \frac{1}{2}$

35) $x - \frac{x}{2} - \frac{x}{3} = 1$

****

5. 2 Quadratic Equations

The standard form of a **quadratic equation** is $\boldsymbol{ax^2 + bx + c = 0}$, where a, b and c are real numbers with $a \neq 0$. The degree of a quadratic equation is 2. A quadratic equation may have either one or two or no real solutions.

Eg: $x^2 - 2x + 3 = 0$; $2y^2 = 1 + 4y - y^2$; $t^2 + 1 = 0$

Zero –Product Property
AB = 0 ***if and only if*** ***A=0*** ***or*** ***B=0***

1) Solving a quadratic Equation by factoring

Solving the quadratic equation when b = 0

***Example* 1:** Solve the quadratic equation $x^2 - 25 = 0$

Solution: We have, $x^2 - 25 = 0$

$\Rightarrow$ $(x - 5)(x + 5) = 0$

$\Rightarrow$ $x - 5 = 0$ or $x + 5 = 0$

$\Rightarrow$ $x = 5$ or $x = -5$

The solutions are **+5** and **-5.**

Solving the quadratic equation when c = 0

***Example* 2:** Solve the quadratic equation $x^2 - 5x = 0$

Solution: We have, $x^2 - 5x = 0$

$\Rightarrow$ $x(x - 5) = 0$

$\Rightarrow$ $x = 0$ or $x - 5 = 0$

$\Rightarrow$ $x = 0$ or $x = 5$

Thus, the solutions are **0** and **5.**

Solving the quadratic equation when b ≠ 0 and c ≠ 0.

***Example* 3:** Solve the equation $x^2 + 5x = 24$

Solution: Consider, $x^2 + 5x = 24$

$\Rightarrow \quad x^2 + 5x - 24 = 0$

Converting to the standard form.

$\Rightarrow \quad (x - 3)(x + 8) = 0$

Factoring the quadratic expression.

$\Rightarrow \quad x - 3 = 0 \quad \text{or} \quad x + 8 = 0$

$\Rightarrow \quad x = 3 \quad \text{or} \quad x = -8$

The solutions are $x = \mathbf{3}$ and $x = \mathbf{-8}$

2) Solving the equation using quadratic formula

The Quadratic formula
The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$ are given by, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

***Example* 4**: Solve the equation $x^2 - 4x + 3 = 0$

Solution: Here we have, a = 1, b = -4 and c = 3

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(3)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 - 12}}{2} \qquad = \frac{4 \pm \sqrt{4}}{2}$$

$$= \frac{4 \pm 2}{2} \qquad = \frac{4 + 2}{2} \text{ or } \frac{4 - 2}{2}$$

$$= \frac{6}{2} \text{ or } \frac{2}{2} \qquad = 3 \text{ or } 1$$

$\therefore$ The solutions are **3** and **1**

***Example* 5:** Solve the equation $3x^2 - 5x - 1 = 0$

Solution: Here we have, a = 3, b = -5 and c = -1

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-1)}}{2(3)}$$

$$= \frac{5 \pm \sqrt{25+12}}{6} = \frac{5 \pm \sqrt{37}}{6}$$

The solutions are $\frac{5+\sqrt{37}}{6}$ and $\frac{5-\sqrt{37}}{6}$

***Example* 6**: Solve the equation $x^2 + 2x + 2 = 0$

Solution: Here we have, a = 1, b = 2 and c = 2

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm \sqrt{-4}}{2}$$

Since, $\sqrt{-4}$ is not real, there are no real solutions for the given equation.

Discriminant and the nature of roots

The value, $\boldsymbol{D = b^2 - 4ac}$ is called the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$.

We can identify the nature of the roots of the equation using the discriminant as given below.

Case 1: If $\boldsymbol{D = b^2 - 4ac > 0}$, then there are two distinct real roots for the equation.

Case 2: If $\boldsymbol{D = b^2 - 4ac = 0}$, then there is only one real root for the equation.

Case 3: If $\boldsymbol{D = b^2 - 4ac < 0}$, then there are no real roots for the equation.

***Example* 7: Evaluate the discriminant of the following quadratic equations and comment on their roots:**

1) $x^2 + 4x - 1 = 0$

Solution: Here a = 1, b = 4 and c = -1
Discriminant, $D = b^2 - 4ac = 4^2 - 4(1)(-1)$
$= 16 + 4 = \mathbf{20}$

Since $b^2 - 4ac = 20 > 0$, the equation has two distinct real roots.

2) $4x^2 - 12x + 9 = 0$

Solution: Here a = 4, b = -12 and c = 9

Discriminant, $D = b^2 - 4ac = (-12)^2 - 4(4)(9)$

$= 144 - 144 = \mathbf{0}$

Since $b^2 - 4ac = 0$, the equation has exactly one real root.

3) $x^2 + 2x = -2$

Solution: We have $x^2 + 2x + 2 = 0$

Here a = 1, b = 2 and c = 2

Discriminant, $D = b^2 - 4ac = 2^2 - 4(1)(2)$

$= 4 - 8 = \mathbf{-4}$

Since $b^2 - 4ac < 0$, the equation has no real roots.

Tutorial - 10

I (1-16) Solve the following quadratic equations for *x*:

1) $x^2 = 64$

2) $3x^2 = 27$

3) $\frac{5}{2}x^2 = \frac{18}{5}$

4) $7x^2 - 52 = 0$

5) $x^2 + 3x - 4 = 0$

6) $x^2 + 8x + 15 = 0$

7) $x^2 + 30x + 200 = 0$

8) $2x^2 - 7x + 6 = 0$

9) $5x^2 - 8x = 4$

10) $6x - 1 = x^2$

11) $(2x+1)^2 = 6$

12) $x^2 = 3(x - 1)$

13) Solve for *y*: $5y^2 - 4y - 2 = 0$

14) Solve for *x*: $x(x - 2) + 3x = 12$

15) Solve for *x*: $\frac{x}{3} = \frac{4}{x-5}$

16) Solve for *x*: $\frac{x+5}{3} = \frac{4}{x-3}$

II (17-21) Obtain the discriminant of the following equations and comment on the nature of the roots:

17) $x^2 - 10x + 16 = 0$

18) $4x^2 + 4x + 1 = 0$

19) $3x^2 + 5x + 4 = 0$

20) $x(x - 3) = 54$

21) $3x^2 + 5x = 2$

****

5. 3 Modeling with Equations

Modeling with equation: It is the process of solving a real life problem by converting the situations into mathematical equations.

Guidelines for modeling with equations

Define variable: Identify the quantity that problem asks you to find.

Set up the model: Formulate the problem as a mathematical expression involving the variable.

Solve the equation: Solve the for the required variable.

Conclusion: Express the solutions as a sentence that answers the question posed in the problem

Modeling with Linear Equations

***Example* 1 :** The perimeter of a fenced rectangular area is 45 m. Write a linear model relating the length and width. If the length is 10 m, find the width.

Solution:

We know that the perimeter p of a rectangle is given by,

$$p = 2l + 2w.$$

So, $45 = 2l + 2w$ is the required linear equation.

Substitute $l = 10$.

$\Rightarrow$ $45 = 2(10) + 2w$

$\Rightarrow$ $45 = 20 + 2w$

$\Rightarrow$ $2w = 45-20$

$\Rightarrow$ $w = 25 \div 2$

$\therefore$ The required width is **12.5 meters**.

Modeling with Quadratic Equations

***Example* 2:** Two consecutive positive odd integers are such that the square of the smaller is eleven more than two times the larger. Find the integers.

Solution:

Let the two consecutive odd integers be $\boldsymbol{x}$ and $\boldsymbol{x + 2}$.

We know that eleven more than two times the larger is $2(x+2)+11$ and the square of the smaller is x^2.

Thus the required equation is

$$x^2 = 2(x+2)+11$$

$\Rightarrow$ $x^2 = 2x + 15$

$\Rightarrow$ $x^2 - 2x - 15 = 0$

$\Rightarrow$ $(x-5)(x+3) = 0$

$\Rightarrow$ $x = 5 \quad x = -3$

Since -3 is not a positive integer, this can’t be part of the solution. So the solution is that $x = 5$ and $x + 2 = 5+2 = 7$

Hence, the required consecutive positive odd integers are **5** and **7**.

Tutorial - 11

I (1-7) answer the following questions using linear equations:

1) Find three consecutive integers whose sum is 27
2) Find four consecutive odd integers whose sum is 96
3) The sum of 40 and twice a number is 132. What is the number?
4) Six more than half of a number is 30. What is the number?
5) Find three consecutive integers with a sum of 63.
6) Find four consecutive even integers with a sum of 60.
7) A rectangular garden is 25 m wide .If its area is 1125m ,what is the length of the garden ?

II (8-12) Answer the following questions using quadratic equations:

8) The product of two consecutive negative integers is 1122. What are the numbers?
9) One number is the square of another. Their sum is 132. Find the numbers.
10) The difference of two numbers is 2 and their product is 224. Find the numbers.
11) Length of a rectangular garden is 3 meters more than the width. If the area of the garden is 550 square meters, find its length and width.
12) The area of a rectangle is 560 square inches. The length is 3 more than twice the width. Find the length and the width.

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CHAPTER 6 – INEQUALITIES

Learning Objective:

- ✓ *Demonstrate an understanding of inequalities and solving linear and quadratic inequalities.*

In the previous chapter, we have studied about equations. In fact, an equation is a formula of the form A=B, where A and B are expressions. In some problems, we may face inequalities instead of equations. For example in your class, the marks of all students in a quiz may be **greater than or equal to** 5.

An **inequality** is a relation that holds between two values when they are different.

An inequality just looks like an equation, the only difference is that the equal to sign is just replaced by one of the symbols $<, >, \leq, \geq$ or $\neq$.

Here are some examples of inequalities;

$5 < 10,$ $\quad 2y \geq -1$ $\quad x \neq 5$ $\quad x < 0$

$3y + 2 > 10$ $\quad 3x^2 + 2x - 5 \geq 0$ $\quad 5x^3 - 4x < 8$

Solution of an Inequality

To solve an inequality that contains a variable means to find all possible values of the variable that make the inequality true. An inequality generally has infinitely many solutions. The solution of a linear inequality can be written as an interval or a union of intervals. The following are some important instructions while you solve inequalities:

Safe Things to Do

You can do these things **without affecting** the direction of the inequality:

- Add (or subtract) a number on both sides
- Multiply (or divide) both sides by a **positive** number
- Simplify a side

***Example* 1:**

Given $x < 7$

$x+2 < 7+2$ (Adding 2 on both sides)

$x+2 < 9$ (simplify a side)

$2(x+2) < 2\times 9$ (multiplied by a positive number)

$2x+4 < 18$ (Simplify a side)

However, these things will change the direction of the inequality

- Multiply (or divide) both sides by a **negative** number
- Swapping left and right hand sides
- Taking reciprocals on both sides (provided both sides are of the same sign)

***Example* 2:**

Given $x+2<5 \Rightarrow \quad 5>x+2$ (Swapping left and right hand sides)

***Example* 3:**

Given $x+2<5 \Rightarrow -2(x+2) > -2\times 5$ (Multiplied by -2)

$\Rightarrow -2x-4 > -10$

***Example* 4:**

Given $x \le 3 \quad \Rightarrow \quad \frac{1}{x} \ge \frac{1}{3}$

6.1 Linear Inequalities

An inequality, which involves linear algebraic expressions, is called a **linear inequality.**

Examples: $x \le 2$, $2x+9 \ge 0$, $3y \le \frac{1}{5}$

Solution of a linear inequality

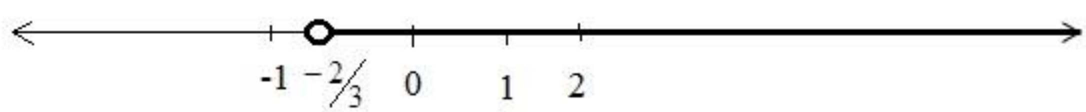
***Example* 5:**
Solve the linear inequality $2x < 8x + 4$ and sketch the solution set.

Solution: Given $2x < 8x + 4$

$$\Rightarrow 2x - 8x < 4$$
$$\Rightarrow -6x < 4$$
$$\Rightarrow \frac{-6x}{-6} > \frac{4}{-6}$$
$$\Rightarrow x > -\frac{2}{3}$$

i.e, the solution set is $\left(\frac{-2}{3}, \infty\right)$.

The solution set can be expressed on a number line as given below:



***Example* 6:**

Solve the inequalities $5 \le 4x - 3 < 17$ and represent the solution graphically.

Solution: Given $5 \le 4x - 3 < 17$

$$\Rightarrow 8 \le 4x < 20$$
$$\Rightarrow \frac{8}{4} \le \frac{4x}{4} < \frac{20}{4}$$
$$\Rightarrow 2 \le x < 5$$

Therefore, the solution set is **[2, 5).**

The solution set can be expressed on a number line as given below:

Tutorial - 12

I 1(i-iii) Choose the correct answer for the following:

1) i) Solve: $-4x > -12$

(A) $x > 3$ (B) $x > -3$ (C) $x < 3$ (D) $x < -3$

ii) Solve: $-5y - 7 \le 3$

(A) $y \ge 2$ (B) $y \ge -2$ (C) $y \le 2$ (D) $y \le -2$

iii) Solve: $\frac{x-3}{5} \ge -2$.

(A) $x \ge 5$ (B) $x \ge 7$ (C) $x \ge -7$ (D) $x \le -7$

II (2-15) Solve the following linear inequalities. Express the solution using interval notation and graph the solution set:

2)	$2x-5>3$	3)	$3x+11<5$
4)	$7-x\geq 5$	5)	$5-3x\leq -16$
6)	$0<5-2x$	7)	$\frac{x}{2}\geq 1-x$
8)	$3x+11\leq 6x+8$	9)	$6-x\geq 2x+9$
10)	$\frac{1}{2}x-\frac{2}{3}>2$	11)	$\frac{2}{5}x+1<\frac{1}{5}-2x$
12)	$2\leq x+5\leq 4$	13)	$5\leq 3x-4\leq 14$
14)	$-3<1-2x\leq 5$	15)	$-7\leq -3x+2<5$

****

6.2 Quadratic Inequalities

An inequality, in which the highest power of the variable is two, is called a **quadratic inequality**.

For example, the inequalities $x^2<5$, $3y^2+5\geq 0$, $3z^2+4z\neq 10$, $ax^2+bx+c>0$ are quadratic inequalities.

Solutions of Quadratic inequalities

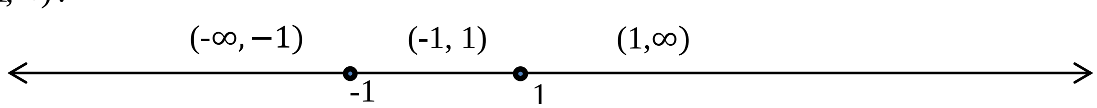
Example 1: Solve the inequality $x^2<1$.

Solution: Given $x^2<1$

$$x^2-1<0$$

$$(x-1)(x+1)<0$$

We know that the corresponding equation $(x-1)(x+1)=0$ has the solutions -1 and 1.As shown in the figure, the numbers-1 and 1 divide the real line into three intervals: $(-\infty,-1),(-1,1)$ and $(1,\infty)$.

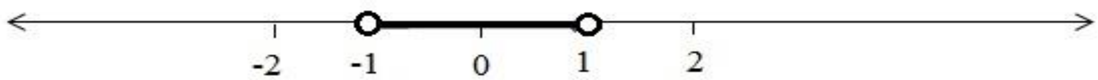


On each of these intervals, we determine the signs of the factors using test values. Select a number inside each interval and check the sign of a particular factor for the selected value.

Now we construct the following sign table.

Interval	$(-\infty,-1)$	(-1, 1)	$(1,\infty)$
Sign of x-1	$-$	$-$	$+$
Sign of x+1	$-$	$+$	$+$
Sign of $(x-1)(x+1)$	$+$	$-$	$+$

From the above table we understand that, $(x-1)(x+1)<0$ on the interval (-1, 1). Graphically the solution is represented as follows:



Example 2: Solve the inequality $x^2-5x+6\leq 0$ and graph the solution on the number line.

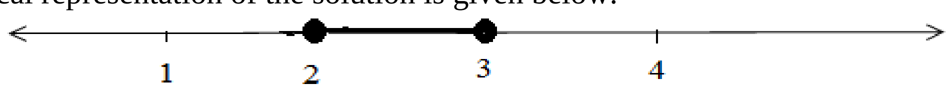
Solution: Given $x^2-5x+6\leq 0$

$(x-2)(x-3)\leq 0$

Interval	$(-\infty, 2)$	(2, 3)	$(3, \infty)$
Sign of $x-2$	$-$	$+$	$+$
Sign of $x-3$	$-$	$-$	$+$
Sign of $(x-2)(x-3)$	$+$	$-$	$+$

From the table we can see that $(x-2)(x-3)\leq 0$ on the interval [2,3].

The graphical representation of the solution is given below.



Tutorial - 13

I (1-10) Solve the quadratic inequalities. Express the solution using interval notation and graph the solution set.

1) $(x+2)(x+3)<0$

2) $(x-5)(x-4)\geq 0$

3) $x(2x+7)\geq 0$

4) $x^2\geq 4$

5) $x^2+3x-18\leq 0$

6) $x^2>3(x+6)$

7) $x^2+2x\geq -1$

8) $x^2<x+2$

9) $3x^2-3x<2x^2+4$

10) $x^2-13x+36\leq 0$

****

CHAPTER 7 - COORDINATE GEOMETRY

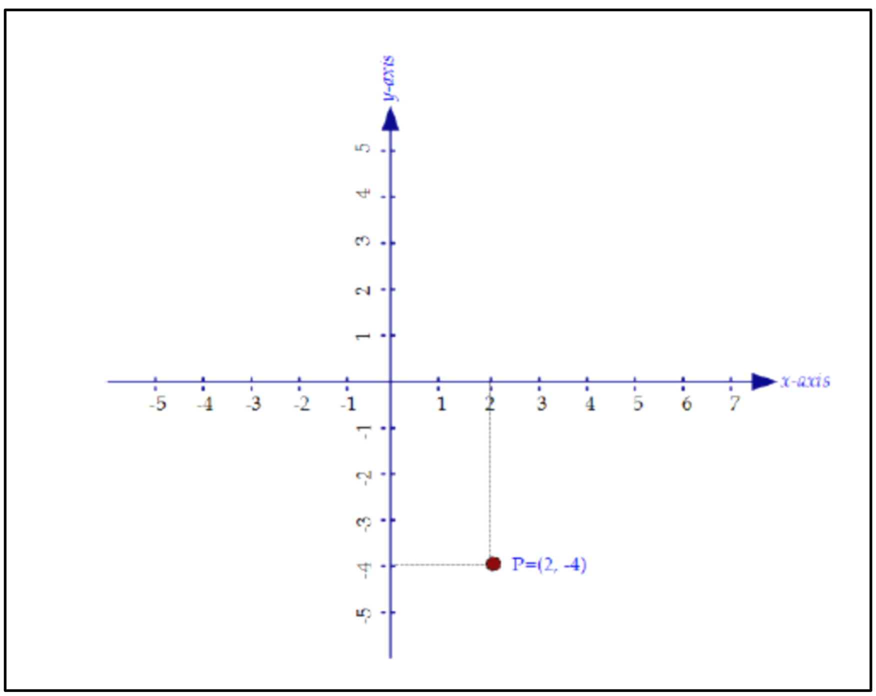
Learning Objectives:

- ✓ *Demonstrate an understanding of Coordinate Plane and the related terms .*
- ✓ *Use coordinate plane to solve algebraic and geometric problems, and understand geometric concepts such as parabolas, circles , perpendicular and parallel lines*

7.1 Introduction to Coordinate Geometry

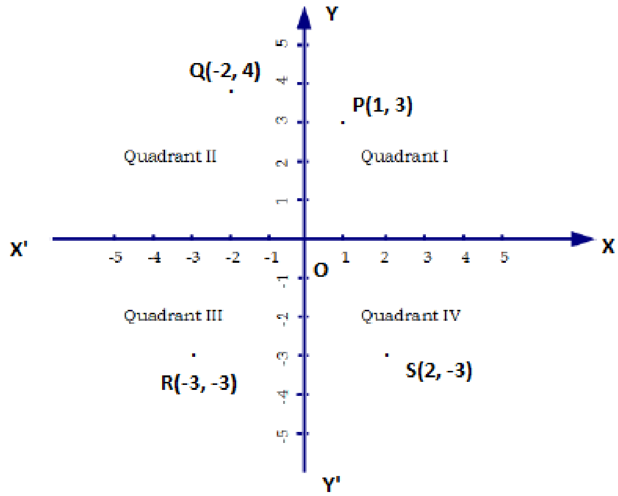
A plane containing two perpendicular number lines, one horizontal and the other vertical, meeting each other at 0 is called a **Cartesian plane** or **coordinate plane.** We use these two lines to locate the exact position of each point in the plane. The horizontal number line is called the **x–axis** and the vertical number line is called the **y – axis.** The x – axis is denoted by X'OX and the y – axis is denoted by Y'OY as give in the figure below.

Each point P on the plane is associated to an ordered pair of values in the form (x, y) where x and y are the perpendicular distances drawn from the point to the y – axis and x – axis respectively. The first value in the pair is called the **x coordinate** or the **abscissa** and the second value is called the **y coordinate** or the **ordinate.** The x coordinate of any point on the y axis is 0 and the y coordinate of any point on the x axis is 0.



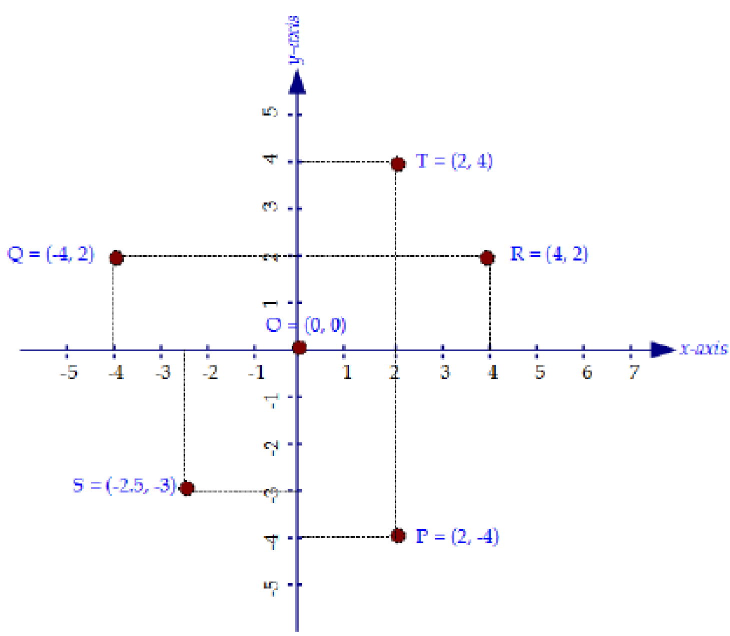
The point where the two axes meet each other is called the **origin.** The origin is always denoted by O(0, 0).

The two axes divide a Cartesian plane into four regions called the **quadrants** namely, the first (I), second (II), third (III) and the fourth (IV) quadrants. Each point in the first quadrant will be of the form (+, +), in II quadrant, it is (-, +), in III it is (-, -) and in IV quadrant it is (+, -).



Thus the point P(1, 3) is in I quadrant, Q(-2, 4) in the II quadrant, R(-3, -3) in the III quadrant and S(2, -3) in the IV quadrant.

Example 1**:** Plot the points P(2, -4), Q(-4, 2), R(4, 2), S(-2.5, -3), T(2, 4) and O(0, 0) on a Cartesian plane.



Graphing Regions in a Coordinate Plane

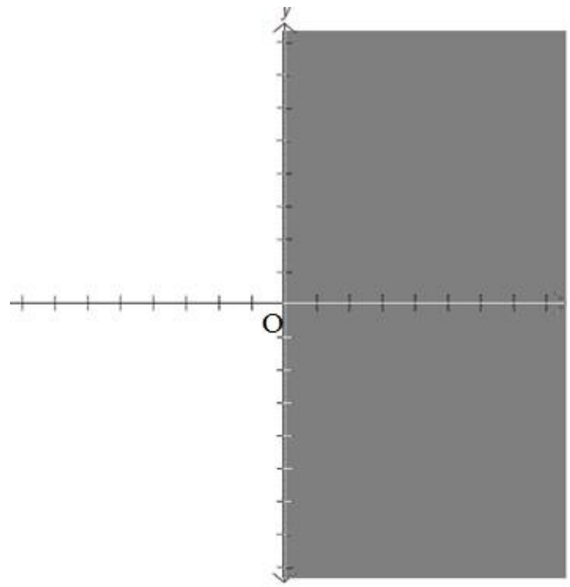
Example 2: Sketch the regions

1) $\{(x,y) : x \geq 0\}$

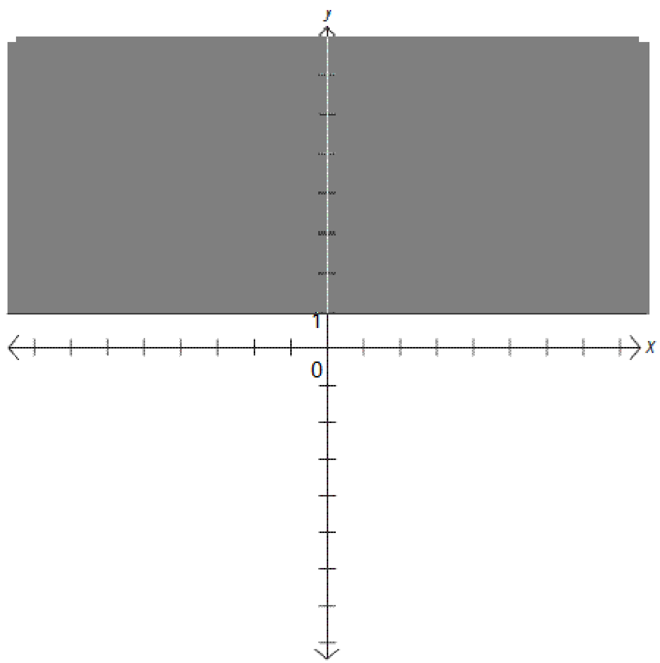
2) $\{(x,y) : y \geq 1\}$

Answer:

1) The shaded portion is the required region.



2) The shaded portion is the required region

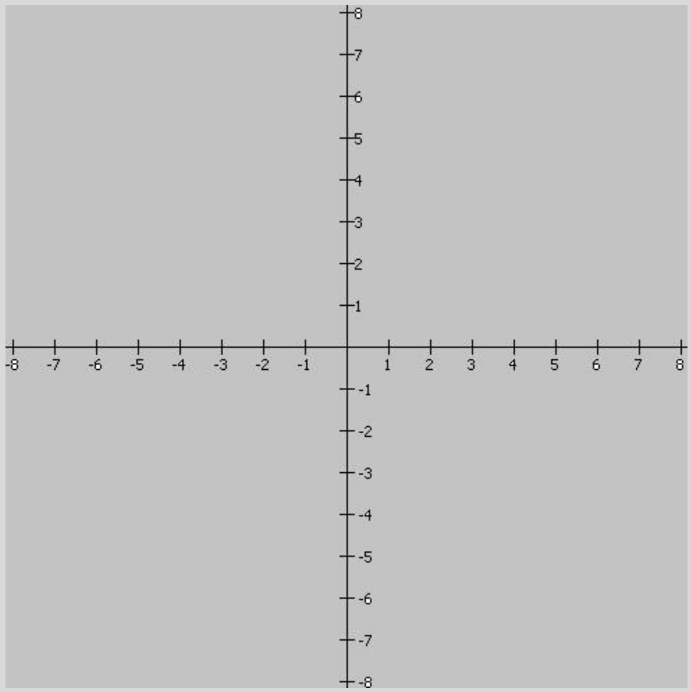


Tutorial - 14

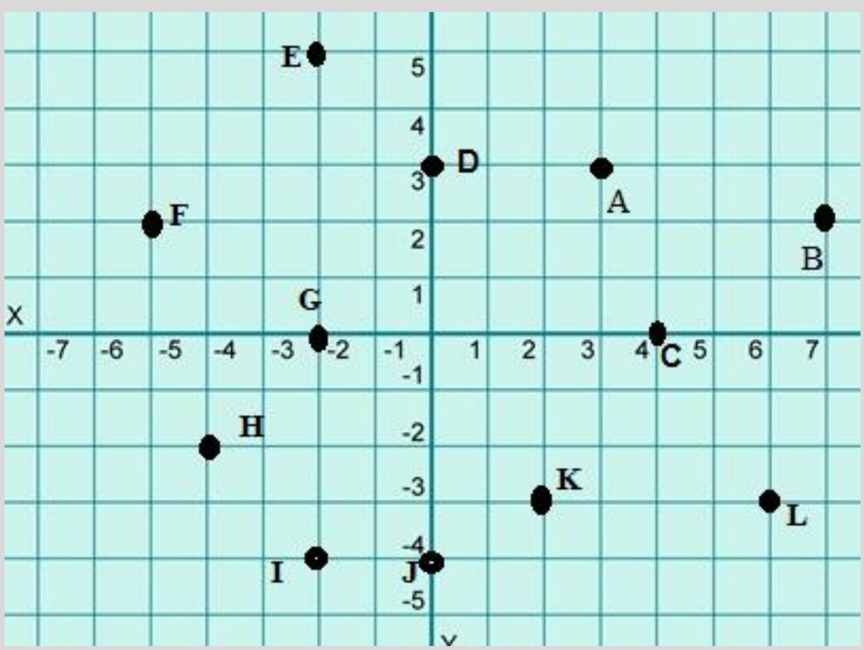
I **(1) Plot the following points on the Cartesian plane:**

O(0, 0), A(2, 3), B(-1, 5), C(6, -4), D(-3, -3), E(5, 0),

F(0, -3.5), G(-2, 0), H(0, 6), I(1.5, -2.5), J(π , $\sqrt{10}$), K(- $\sqrt{8}$, $\sqrt{8}$)

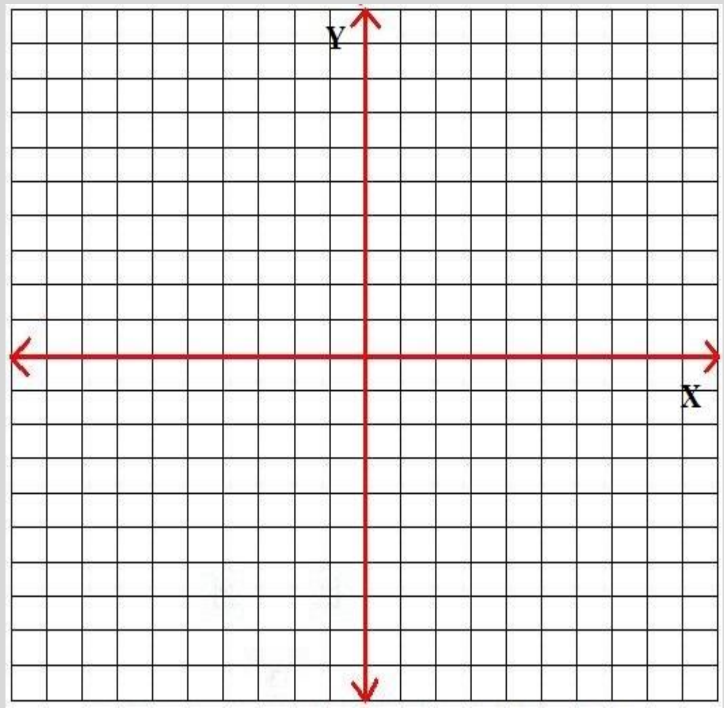


II **(2) Find the coordinates of the points shown in the figure**

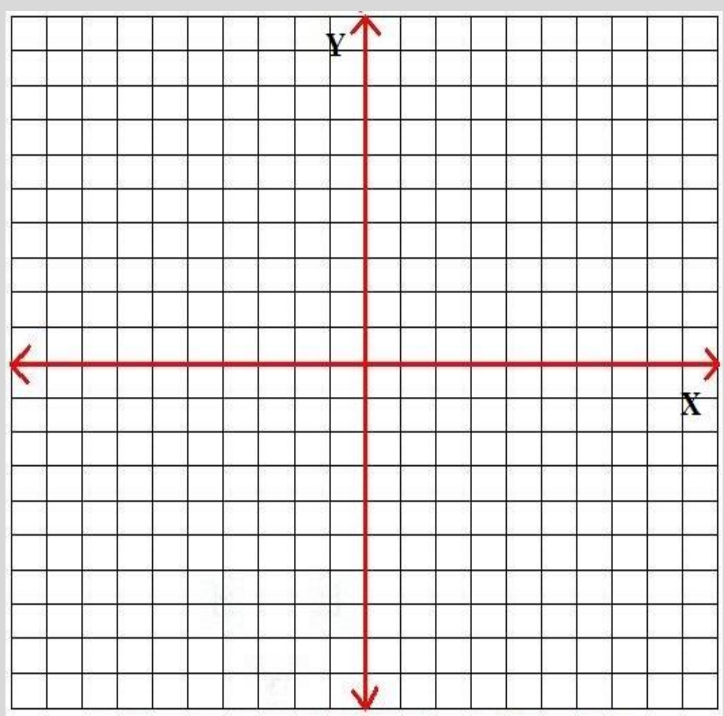


III (3-7) Sketch the given regions:

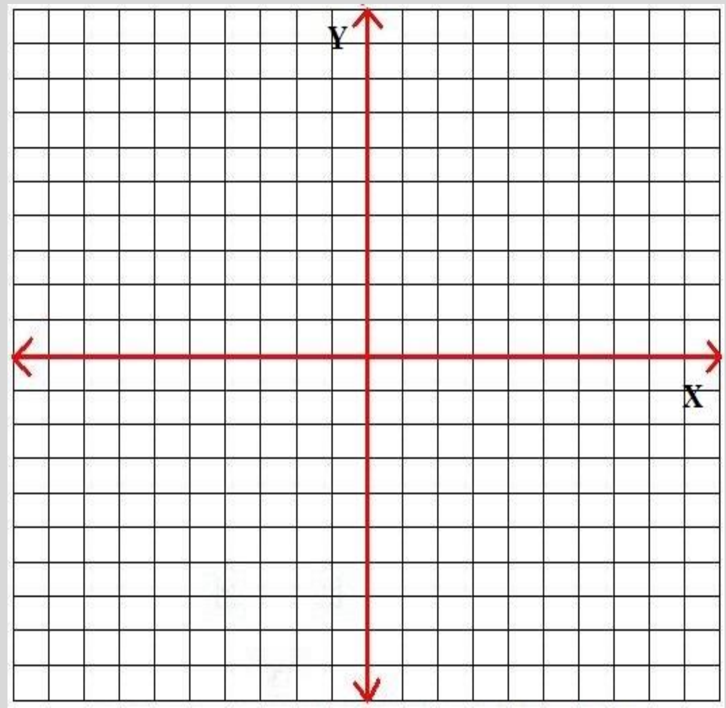
3) {(x,y): y < 0}



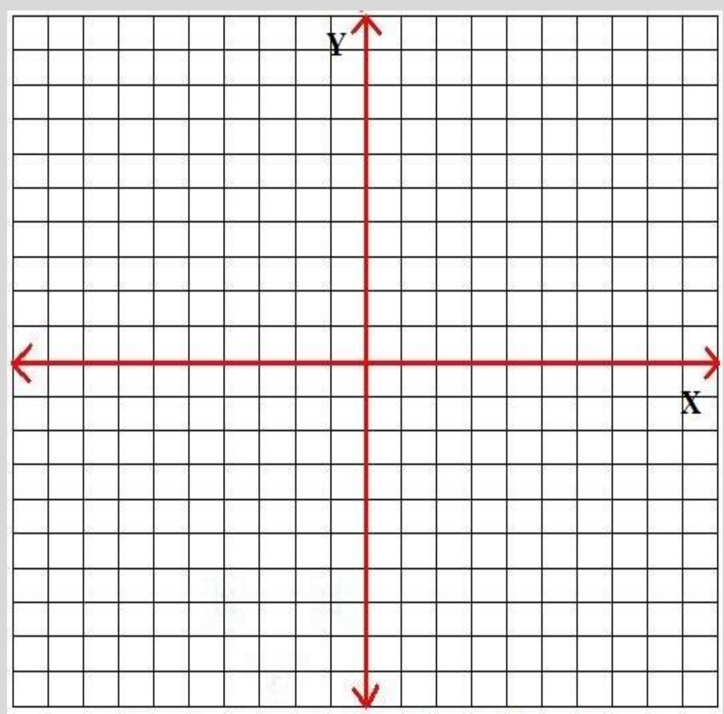
4) {(x,y) : x = 1}



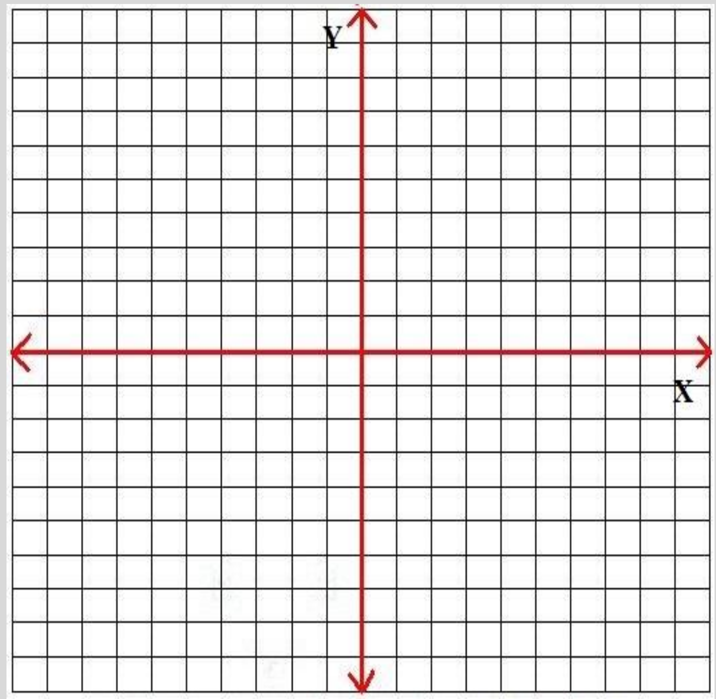
5) $\{(x,y) : x \geq 2\}$



6) $\{(x,y) : 1 < x < 2\}$



7) $\{(x,y) : 0 \leq y \leq 5\}$



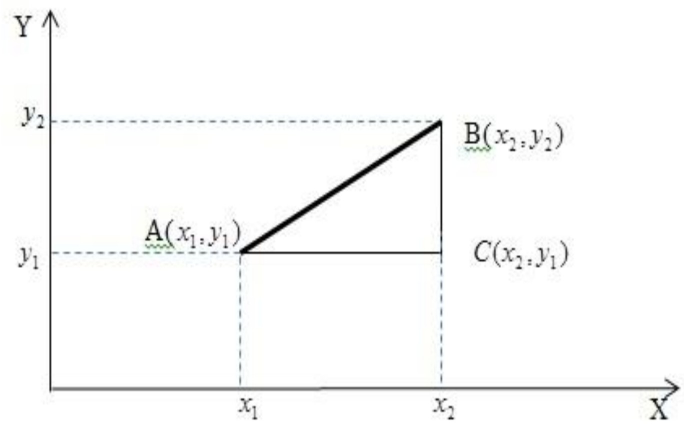
IV (8) Fill in the following table with the quadrants in which the given points lie.

No.	*Point*	*Quadrant*
1	A(2, 3)	
2	B(-1, 5)	
3	C(6, -4)	
4	D(-3, -3)	
5	E(5, 0)	
6	F(0, -3.5)	
7	G(-2, 0)	
8	H(0, 6)	

****

7.2 DISTANCE AND MIDPOINT FORMULA

Let us find the distance between two points $A(x_1, y_1)$ and B (x_2, y_2) in the plane.



From the above figure,
$d(A,C) = |x_2 - x_1|$ and $d(B,C) = |y_2 - y_1|$.
Since triangle ABC is a right-angled triangle, by Pythagorean Theorem, we have

$$d(A,B) = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The distance between the points $A(x_1, y_1)$ and B (x_2, y_2) is given by

$$d(A,B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note: The distance between a point A(x,y) and the Origin is $\sqrt{x^2 + y^2}$.
***Example* 1:** Find the distance between P(-2, 3) and Q(1, -3).

Solution: Here $x_1 = -2$, $y_1 = 3$, $x_2 = 1$ and $y_2 = -3$.

Then the distance is given by,

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(1 - (-2))^2 + (-3 - 3)^2} \\ &= \sqrt{9 + 36} \\ &= 3\sqrt{5} \quad \text{units} \end{aligned}$$

***Example* 2:** Find the distance between the points (4,3) and the origin.

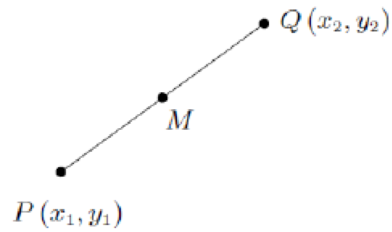
Solution: Here $d = \sqrt{x^2 + y^2} = \sqrt{4^2 + 3^2}$
$= \sqrt{25} =$ **5 units**

Midpoint Formula

Consider two points P(x_1, y_1) and Q(x_2, y_2) in a plane. Then the midpoint M of the line segment is the point on PQ such that PM = QM

Clearly, the coordinates of M are given by,

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$



***Example* 3:** Find the midpoint of the line segment connecting P(-2, 3) and Q(1, -3).

Solution: Here $x_1 = -2$, $y_1 = 3$, $x_2 = 1$ and $y_2 = -3$.

Then the midpoint M is given by,

$$\begin{aligned} M &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{(-2) + 1}{2}, \frac{3 + (-3)}{2}\right) \\ &= \left(-\frac{1}{2}, \frac{0}{2}\right) \\ &= \left(-\frac{1}{2}, 0\right) \end{aligned}$$

Thus the midpoint is $\left(-\frac{1}{2}, 0\right)$.

***Example* 4:** *A quadrilateral whose diagonals bisect each other is called a parallelogram*. Using this information, show that the quadrilateral with vertices A(-1,2) , B(3,0) , C(5,6) and D(1,8) is a parallelogram.

Solution: The midpoint of the diagonal AC is $(\frac{-1+5}{2}, \frac{2+6}{2}) = (2,4)$.

The midpoint of the diagonal BD is $(\frac{3+1}{2}, \frac{0+8}{2}) = (2,4)$.

Since the diagonals bisect each other, ABCD is a parallelogram.

Tutorial - 15

I (1-5) Calculate the distance between the following pairs of points.

1) (2, 3) and (5, 7)

2) (-1, 2) and (3, -10)

3) $(\frac{1}{2}, 4)$ and $(\frac{3}{2}, -1)$

4) $(\frac{-2}{3}, \frac{3}{2})$ and $(\frac{7}{3}, 2)$

5) (0, 0) and (x, y)

II (6-10) Find the midpoint of the line segments joining the following pair of points:

6) (2, 3) and (5, 7)

7) (-1, 2) and (3, -10)

8) (0, 4) and (0, -1)

9) (5, 0) and (-3, 0)

10) (0, 0) and (x, y)

III (11-16) Answer the following word problems:

11) Which of the points A(5,4) or B (-3,8) closer to the origin?

12) Find the length of the line segment whose endpoints are (-3, 4) and (5,4).

13) Find the distance between the origin and the midpoint of the line segment joining A(10, 8) and B(2,8).

14) Find the distance from the point P(2, - 3) to the midpoint of the line segment joining A(6, 1) and B(- 4, 7).

15) Two places A and B are located by the points (3, -2) and (-3, 6) respectively. A man is walking from A to B with a speed of 2m/s. How much time he will take to reach B? (*Distance between A and B is measured in metre.*)

16) An ant travels a distance PQ in 3 minutes. If the points P and Q are given by (5, 10) and (- 4, - 2) respectively, calculate the speed of the ant. (*Distance between P and Q is measured in metre.*)

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CHAPTER 8 - LINES

Learning Objective:

- ✓ *Use coordinate plane to solve algebraic and geometric problem, and understand geometric concepts such as equation of a line, perpendicular and parallel lines.*

In this chapter, we shall deal with the slope of a straight line and the equation of a straight line in different forms. We shall also discuss the concept of parallel and perpendicular lines.

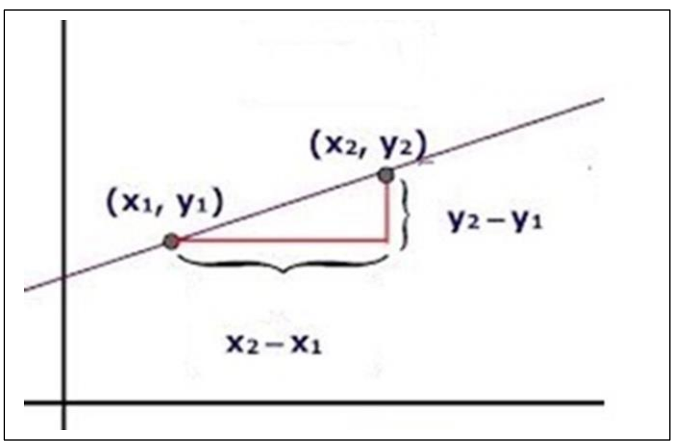
8.1 Slope and Equation of a straight line

Slope of a straight line

The term **slope** refers to the steepness of a straight line. It measures rate at which the straight line rises or falls.

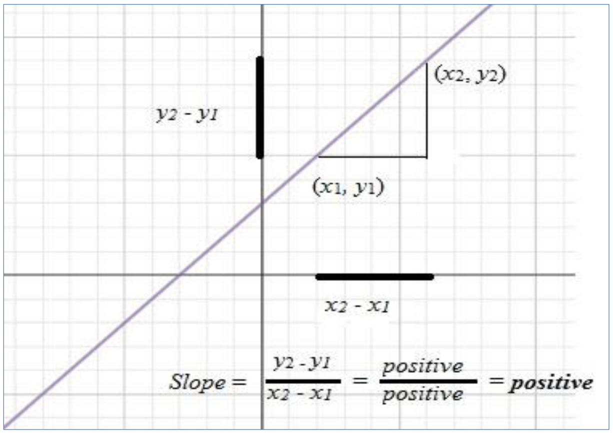
We shall define the slope of a straight line as given below:
Consider a straight line *l* passing through two given points, say (x_1, y_1) and (x_2, y_2). Then the slope of line *l* is given by the ratio of change in y coordinates to the change in *x* coordinate.
i.e,

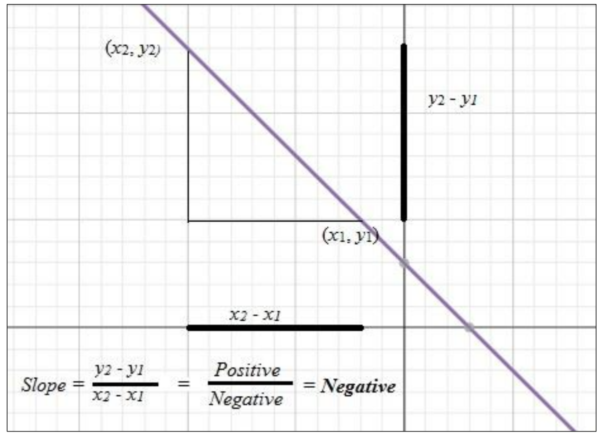
$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$



The slope of a straight line can also be defined as the tangent of the angle between the x-axis and the line. The slope of a horizontal line is zero whereas the slope of a vertical line is not defined. (*Why*?)

The slope of a straight line can be positive or negative as given in the following diagrams. The lines with positive slope slant upward to the right and the lines with negative slope slant downward to the right.





***Example* 1: Finding the slope of a straight line through two given points**

Find the slope of the straight line that passes through the points P(2, -1) and Q(-3, - 3).

Solution: Here, $x_1 = 2$, $y_1 = -1$, $x_2 = -3$ and $y_2 = -3$.

By definition,

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-3)-(-1)}{(-3)-(2)}$$
$$= \frac{-2}{-5} = \frac{2}{5}$$

Thus the required slope of the line is $\frac{2}{5}$.

***Example* 2:**

Find the slope of the straight line that passes through the points A(2, 3) and the origin.

Solution: Here, $x_1 = 2$, $y_1 = 3$, $x_2 = 0$ and $y_2 = 0$.

By definition,

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-3}{0-2}$$
$$= \frac{-3}{-2} = \frac{3}{2}$$

Thus the required slope of the line is $\frac{3}{2}$.

Equation of a Line

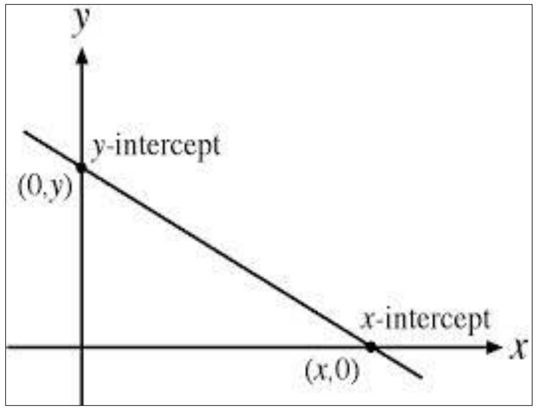
The equation of a line or any curve is the general rule satisfied by the x and y coordinates of the points on the curve. On the other hand the collection (*locus*) of all points satisfying an equation involving x and/or y represents a curve in the Coordinate plane.

The equation of a straight line will always be linear.

Intercepts of a Line

The x-coordinate of the point where a straight line intersects the x-axis is called the **x-intercept** of the line and it can be obtained by setting $y = 0$ in the equation of the line.

Similarly, the y-coordinate of the point where a straight line intersects the y-axis is called the **y – intercept** of the line and it can be obtained by setting $x = 0$ in the equation of the line.



Slope – intercept form of a Line

Consider a straight line with ***y intercept b and a given slope m***.
Let (x, y) be any point on the line.
Since the y-intercept of the line is known to us, the line passes through $(0, b)$.

Thus, by the definition of slope of a line,

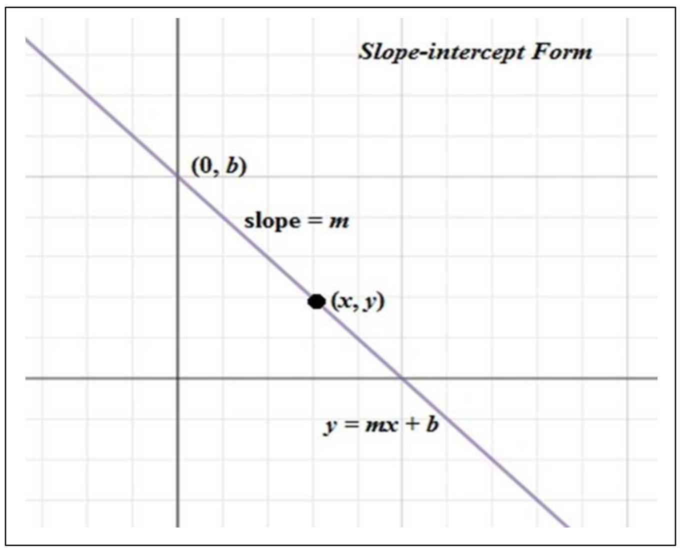
$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow \quad m = \frac{y - b}{x - 0}$$

$$\Rightarrow \quad m = \frac{y - b}{x}$$

$$\Rightarrow \quad mx = y - b$$

$$\Rightarrow \quad y = mx + b$$



Thus, the slope intercept form of a straight line is given by

$$\boldsymbol{y = mx + b}$$

General Equation of a Line

The general linear equation in two variables x and y given by, $\boldsymbol{Ax + By + C = 0}$ (where A and B are not simultaneously zero) represents a straight line.
Conversely, the equation of every line is a linear equation in two variables.

Thus the general equation of a line is given by

$$\boldsymbol{Ax + By + C = 0}$$

where at least one of the coefficients A and B is non-zero.

***Example* 3:**
Find the equation of a line with slope – 2 and y-intercept 3.

Solution: Here, $m = -2$ and $b = 3$.
Substituting these values in the slope intercept form of a line we get,

$$y = mx + b$$

$$\Rightarrow \quad y = -2x + 3$$

$$\Rightarrow \quad 2x + y - 3 = 0$$

Hence, the required equation of the line is $\mathbf{2x + y - 3 = 0}$.

***Example* 4:**
Find the slope and y-intercept of the line $2x - 3y - 3 = 0$.

Solution:

(**Method 1)** Consider the equation,

$$2x - 3y - 3 = 0$$

$$\Rightarrow \quad -3y = -2x + 3$$

$$\Rightarrow \quad y = \frac{-2}{-3}x + \frac{3}{-3}$$

$$\Rightarrow \quad y = \frac{2}{3}x - 1$$

Upon comparing with the slope intercept form, $y = mx + b$, we get
Slope of the line = $m = \frac{2}{3}$ and the y-intercept = $b = -1$.

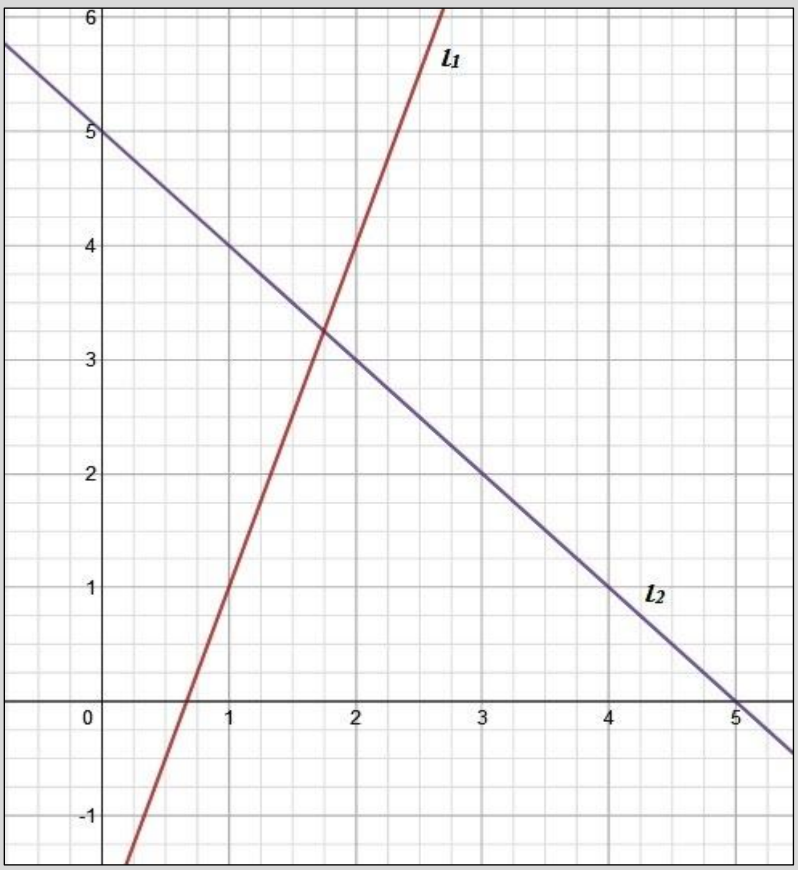
***Note*:**
If you need only the y-intercept of the line, you may set $x = 0$ in the equation and solve for y.
i.e, $2(0) - 3y - 3 = 0 \Rightarrow -3y - 3 = 0$
$\Rightarrow -3y = 3$
$\Rightarrow y = -1$.

Tutorial – 16

I (1-8) Find the slope of the line through P and Q:

1) P(0, 0) and Q(4, -2)
2) P(0, 0) and Q(-2, -6)
3) P(5, 3) and Q(10, -1)
4) P(1, 3) and Q(4, 4)
5) P(4, 2) and Q(2, 4)
6) P(-4, 3) and Q(2, -5)
7) P(1, -3) and Q(-1, 6)
8) P(-1, -2) and Q(-3, -6)

II (9) Find the slope of lines l_1 and l_2 given in the following figure.



III (10-15) Answer the following questions:

10) Find the equation of a line whose slope is 3 and y – intercept – 1.

11) Find the general equation of a line whose slope is - 2 and y – intercept 3.

12) Obtain the equation of a line whose slope is $\frac{5}{2}$ and y – intercept $-\frac{1}{2}$.

13) Find the general equation of a line whose slope is $-\frac{1}{3}$ and y – intercept $\frac{4}{3}$.

14) Obtain the general equation of a line whose slope is $\frac{2}{3}$ and y – intercept - 4

15) Obtain the equation of a line whose slope is $-\frac{3}{2}$ and y – intercept 1.

IV (16-22) Find the slope and *y* –intercept of the given lines:

16) $x + y = 1$

17) $3x - 2y = 6$

18) $4x - y = 0$

19) $-3x - 2y = 1$

20) $3y - 2x = 12$

21) $3x - 4y - 12 = 0$

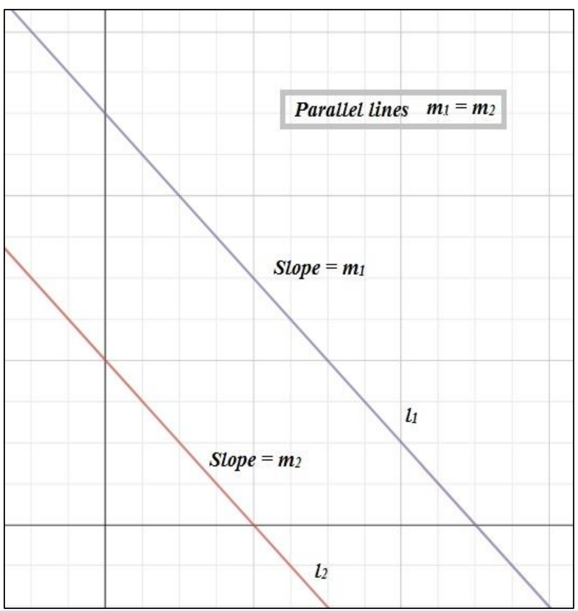
22) $2y = 3$

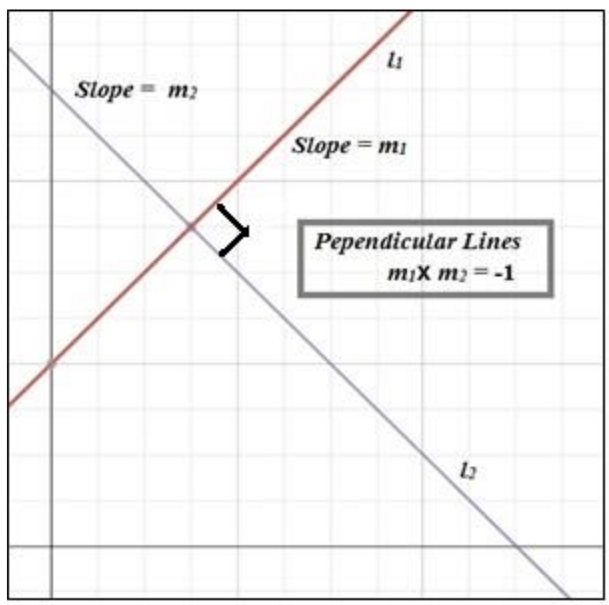
****

8.2 Parallel and Perpendicular Lines

Two or more lines that are equidistant from each other are called **parallel** lines and two lines with an angle between them is a right angle (90^0) are said to be **perpendicular lines.**

Two lines are parallel if and only if their slopes are equal. In a similar way, two lines are perpendicular if and only if the product of their slopes is – 1.





Consider two lines l_1 and l_2 whose slopes are m_1 and m_2 respectively.

Then, l_1 is parallel to l_2 if and only if $\boldsymbol{m_1 = m_2}$

and l_1 is perpendicular to l_2 if and only if $\boldsymbol{m_1 \times m_2 = -1}$

***Example* 1:**

Check whether the lines represented by the following pairs of equations are parallel or perpendicular.

(i) $2x - y = 3$ and $4x - 2y - 1 = 0$
(ii) $3x - 2y = 1$ and $2x + 3y + 5 = 0$
(ii) $x + 2y - 2 = 0$ and $2x + y + 1 = 0$

Solution:

(i) Consider the line, $2x - y = 3 \Rightarrow y = 2x - 3$. $\therefore$ Slope $m_1 = 2$.

Now, consider $4x - 2y - 1 = 0 \Rightarrow y = 2x - \frac{1}{2}$.$\therefore$ Slope $m_2 = 2$.
Since $m_1 = m_2$, the given lines are parallel.

(ii) Consider $3x - 2y = 1 \Rightarrow y = \frac{3}{2}x - \frac{1}{2}$. $\therefore$ Slope $m_1 = \frac{3}{2}$
Now, consider $2x + 3y + 5 = 0 \Rightarrow y = -\frac{2}{3}x - \frac{5}{3}$. $\therefore$ Slope $m_2 = -\frac{2}{3}$
Since $m_1 \neq m_2$, the given lines are not parallel.
But we have, $m_1 \times m_2 = \frac{3}{2} \times -\frac{2}{3} = -1$

$\therefore$ The given lines are perpendicular.

(iii) Consider $x + 2y - 2 = 0 \Rightarrow y = -\frac{1}{2}x + 1$. $\therefore$ Slope $m_1 = -\frac{1}{2}$
Now, consider $2x + y + 1 = 0 \Rightarrow y = -2x - 1$. $\therefore$ Slope $m_2 = -2$

Since $m_1 \neq m_2$, the given lines are not parallel.

$m_1 \times m_2 = -\frac{1}{2} \times -2 = 1 \neq -1$. $\therefore$ the given lines are not perpendicular.
Hence, the given pair of lines are neither parallel nor perpendicular.

***Example* 2:**

(i) Find an equation of a line with y-intercept -1 that is parallel to $2x - 3y = 4$.

(ii) Find an equation of a line with y-intercept 3 that is perpendicular to $2x - 3y = 4$.

Solution: Consider the equation $2x – 3y = 4$.

Slope of the line represented by this equation is $m_1 = -\frac{2}{(-3)} = \frac{2}{3}$.

(i) Let the slope of the required line be m. Since this line is parallel to the given line, $m = m_1$.

$$\therefore\ m = \frac{2}{3}.$$

Also, given that the y-intercept of the required line is - 1.

Thus, using the slope intercept form, we get,

$$y = mx + b$$

$$\Rightarrow\quad y = \frac{2}{3}x - 1$$

$$\Rightarrow\quad 3y = 2x - 3$$

$$\Rightarrow\quad 2x - 3y - 3 = 0.$$

Hence, the required equation of the line is $\mathbf{2x - 3y - 3 = 0}$.

(ii) Let the slope of the required line be m. Since this line is perpendicular to the given line,

$$m \times m_1 = -1$$

$$\therefore\ m \times \frac{2}{3} = -1 \Rightarrow m = -\frac{3}{2}$$

Also, given that the y-intercept of the required line is 3.

Thus, using the slope intercept form, we get,

$$y = mx + b$$

$$\Rightarrow\quad y = -\frac{3}{2}x + 3$$

$$\Rightarrow\quad 2y = -3x + 6$$

$$\Rightarrow\quad 3x + 2y - 6 = 0.$$

Hence, the required equation of the line is $\mathbf{3x + 2y - 6 = 0}$.

Vertical and Horizontal Lines

A line whose slope is zero is called a **horizontal line**. It is parallel to the x-axis. A line that does not have a slope is called a **vertical line**. It is parallel to the y-axis

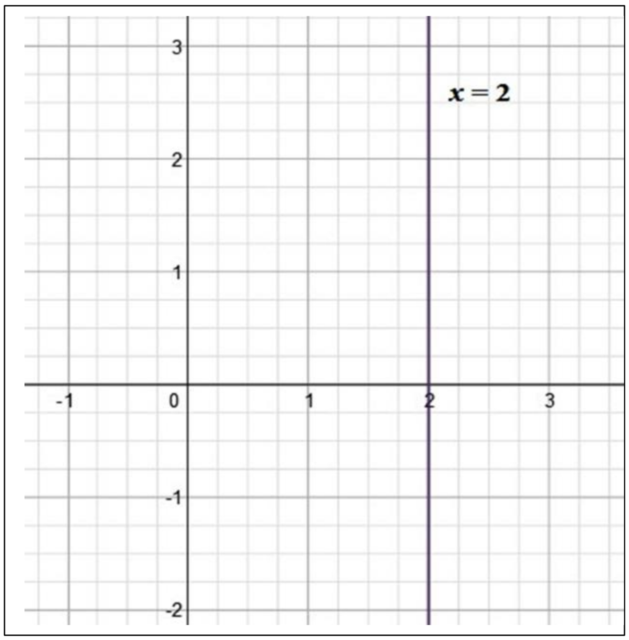
The y-coordinate of every point on a horizontal line is the same whereas the x-coordinate of every point on a vertical line also will be same.

Thus, the equation of a vertical line through (a, b) is $\boldsymbol{x = a}$.

and the equation of a horizontal line through (a, b) is $\boldsymbol{y = b}$.

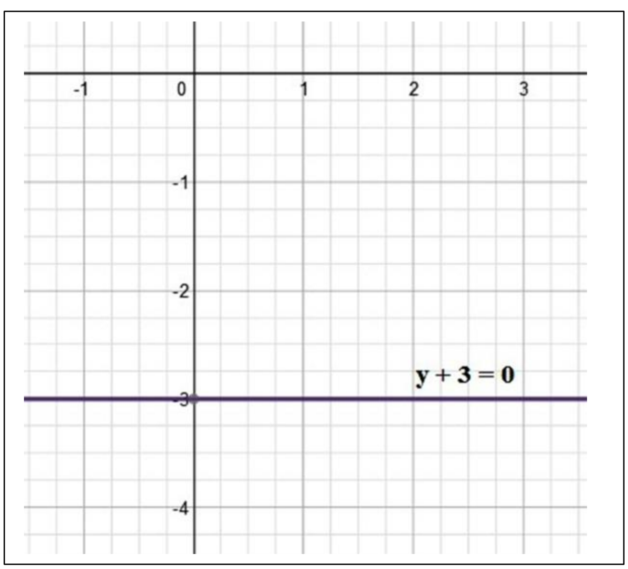
***Example* 3:** Sketch the graphs of the following equations: (i) $x = 2$ (ii) $y + 3 = 0$.

Solution:(i)



The graph of the equation $x = 2$ *is a vertical line passing through* (2, 0).

(ii)



The graph of the equation $y + 3 = 0$ *is a horizontal line passing through* (0, - 3).

Tutorial – 17

I (1 - 8) Answer the following questions:

1) Check whether the straight lines represented by $3x - 2y = 1$ and $x + 3y = 2$ are parallel to each other. Justify your answer.
2) Check whether the straight lines represented by $3x - 2y = 1$ and $6x - 4y - 2 = 0$ are parallel to each other. Give reasons.
3) Check whether the straight lines represented by $x + 3y = 2$ and $3x - y + 3 = 0$ are perpendicular to each other. Justify your answer.
4) Check whether the straight lines represented by $3x - y + 3 = 0$ and $6x - 4y - 2 = 0$ are perpendicular to each other. Justify your answer.
5) Find the equation of the straight line whose y-intercept is -2 and is parallel to the line represented by $2x - y = 3$.
6) Find the equation of the straight line whose y-intercept is 3 and is parallel to the line represented by $3x - 2y +1 = 0$.
7) Find the equation of the straight line whose y-intercept is 3 and is perpendicular to The line represented by $2x - y = 3$.
8) Find the equation of the straight line whose y-intercept is - 2 and is perpendicular to the line represented by $3x - 2y +1 = 0$.

II (9-12) Answer the following questions:

9) Use the slopes to show that A(1, 1), B(7, 4), C(5, 10) and D(-1, 7) are vertices of a parallelogram.

10) Use slopes to show that A(-3, -1), B(3, 3), C(-9, 8) are vertices of a right triangle.

11) Use slopes to show that A(1, 1), B(11, 3), C(10, 8) and D(0, 6) are vertices of a rectangle.

12) Use slopes to determine whether the given points are collinear (*lie on a line*).
(i) (1, 1), (3, 9), (6, 21)
(ii) (-1, 3), (1, 7), (4, 15)

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CHAPTER 9 – FUNCTIONS

Learning Objectives:

- ✓ *Demonstrate understanding of the definition of a function and its graph.*
- ✓ *Identify the domain and range of different functions.*

9.1 Functions and Function Notation

What is a Function?

The natural world is full of relationships between quantities that change. When we see these relationships, it is natural for us to ask "If I know one quantity, can I then determine the other?" This establishes the idea of an input quantity, or independent variable, and a corresponding output quantity, or dependent variable. From this we get the notion of a functional relationship in which the output can be determined from the input.

For some quantities, like height and age, there are certainly relationships between these quantities. Given a specific person and any age, it is easy enough to determine their height, but if we tried to reverse that relationship and determine age from a given height, that would be problematic, since most people maintain the same height for many years.

Function

Function: A rule for a relationship between an input, or independent, quantity and an output, or dependent, quantity in which each input value uniquely determines one output value. We say "the output is a function of the input."

Example 1

In the height and age example above, is height a function of age? Is age a function of height?

In the height and age example above, it would be correct to say that height is a function of age, since each age uniquely determines a height. For example, on my 18^{th} birthday, I had exactly one height of 69 inches.

However, age is not a function of height, since one height input might correspond with more than one output age. For example, for an input height of 70 inches, there is more than one output of age since I was 70 inches at the age of 20 and 21.

Example 2

At a coffee shop, the menu consists of items and their prices. Is price a function of the item? Is the item a function of the price?

We could say that price is a function of the item, since each input of an item has one output of a price corresponding to it. We could not say that item is a function of price, since two items might have the same price.

Example 3

In many classes the overall percentage you earn in the course corresponds to a decimal grade point. Is decimal grade a function of percentage? Is percentage a function of decimal grade?

For any percentage earned, there would be a decimal grade associated, so we could say that the decimal grade is a function of percentage. That is, if you input the percentage, your output would be a decimal grade. Percentage may or may not be a function of decimal grade, depending upon the teacher's grading scheme. With some grading systems, there are a range of percentages that correspond to the same decimal grade.

Function Notation

To simplify writing out expressions and equations involving functions, a simplified notation is often used. We also use descriptive variables to help us remember the meaning of the quantities in the problem.

Rather than write "height is a function of age", we could use the descriptive variable h to represent height and we could use the descriptive variable a to represent age.

"height is a function of age"	if we name the function f we write
"h is f of a"	or more simply
$h = f(a)$	we could instead name the function h and write
$h(a)$	which is read "h of a"

Remember we can use any variable to name the function; the notation $h(a)$ shows us that h depends on a. The value "a" must be put into the function "h" to get a result. Be careful - the parentheses indicate that age is input into the function (Note: do not confuse these parentheses with multiplication!).

Function Notation

The notation output = f(input) defines a function named f. This would be read "output is f of input"

Example 4

Introduce function notation to represent a function that takes as input the name of a month, and gives as output the number of days in that month.

The number of days in a month is a function of the name of the month, so if we name the function *f*, we could write "days = *f*(month)" or $d = f(m)$. If we simply name the function *d*, we could write $d(m)$

For example, *d*(March) = 31, since March has 31 days. The notation $d(m)$ reminds us that the number of days, *d* (the output) is dependent on the name of the month, *m* (the input)

Example 5

A function $N = f(y)$ gives the number of police officers, *N*, in a town in year *y*. What does *f*(2005) = 300 tell us?

When we read *f*(2005) = 300, we see the input quantity is 2005, which is a value for the input quantity of the function, the year (*y*). The output value is 300, the number of police officers (*N*), a value for the output quantity. Remember $N=f(y)$. So this tells us that in the year 2005 there were 300 police officers in the town.

Tables as Functions

Functions can be represented in many ways: Words (as we did in the last few examples), tables of values, graphs, or formulas. Represented as a table, we are presented with a list of input and output values.
In some cases, these values represent everything we know about the relationship, while in other cases the table is simply providing us a few select values from a more complete relationship.

Table 1: This table represents the input, number of the month (January = 1, February = 2, and so on) while the output is the number of days in that month. This represents everything we know about the months & days for a given year (that is not a leap year)

(input) Month number, *m*	1	2	3	4	5	6	7	8	9	10	11	12
(output) Days in month, *D*	31	28	31	30	31	30	31	31	30	31	30	31

Table 2: The table below defines a function $Q = g(n)$. Remember this notation tells us *g* is the name of the function that takes the input *n* and gives the output *Q*.

n	1	2	3	4	5
Q	8	6	7	6	8

Table 3: This table represents the age of children in years and their corresponding heights. This represents just some of the data available for height and ages of children.

(input) *a*, age in years	5	5	6	7	8	9	10
(output) *h*, height inches	40	42	44	47	50	52	54

Example 6

Which of these tables define a function (if any)?

Input	Output
2	1
5	3
8	6

Input	Output
-3	5
0	1
4	5

Input	Output
1	0
5	2
5	4

The first and second tables define functions. In both, each input corresponds to exactly one output. The third table does not define a function since the input value of 5 corresponds with two different output values.

Solving and Evaluating Functions:

When we work with functions, there are two typical things we do: evaluate and solve. Evaluating a function is what we do when we know an input, and use the function to determine the corresponding output. Evaluating will always produce one result, since each input of a function corresponds to exactly one output.

Solving equations involving a function is what we do when we know an output, and use the function to determine the inputs that would produce that output. Solving a function could produce more than one solution, since different inputs can produce the same output.

Example 7

Using the table shown, where $Q=g(n)$

a) Evaluate $g(3)$

n	1	2	3	4	5
Q	8	6	7	6	8

Evaluating $g(3)$ (read: "*g* of 3")
means that we need to determine the output value, Q, of the function g given the input value of $n=3$. Looking at the table, we see the output corresponding to $n=3$ is $Q=7$, allowing us to conclude $g(3) = 7$.

b) Solve $g(n) = 6$

Solving $g(n) = 6$ means we need to determine what input values, n, produce an output value of 6. Looking at the table we see there are two solutions: $n = 2$ and $n = 4$.

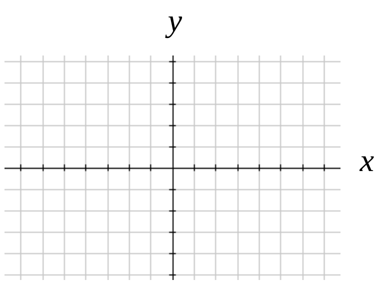
When we input 2 into the function *g*, our output is $Q = 6$

When we input 4 into the function *g*, our output is also $Q = 6$

Graphs as Functions

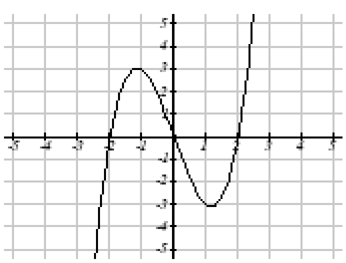
Oftentimes a graph of a relationship can be used to define a function. By convention, graphs are typically created with the input quantity along the horizontal axis and the output quantity along the vertical.

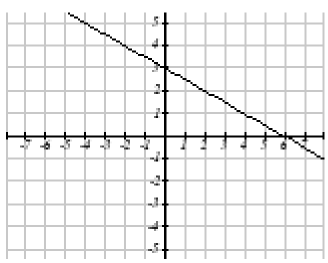
The most common graph has *y* on the vertical axis and *x* on the horizontal axis, and we say *y* is a function of *x*, or $y = f(x)$ when the function is named *f*.

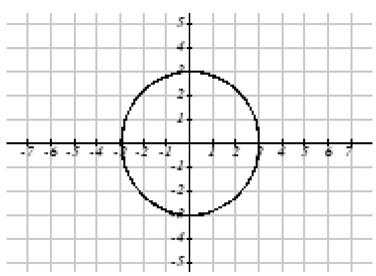


Example 8

Which of these graphs defines a function $y=f(x)$?







Looking at the three graphs above, the first two define a function $y=f(x)$, since for each input value along the horizontal axis there is exactly one output value corresponding, determined by the *y*-value of the graph. The 3rd graph does not define a function $y=f(x)$ since some input values, such as $x=2$, correspond with more than one output value.

Vertical Line Test

The **vertical line test** is a handy way to think about whether a graph defines the vertical output as a function of the horizontal input. Imagine drawing vertical lines through the graph. If any vertical line would cross the graph more than once, then the graph does not define only one vertical output for each horizontal input.

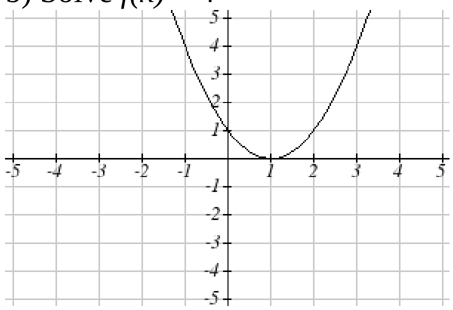
Evaluating a function using a graph requires taking the given input and using the graph to look up the corresponding output. Solving a function equation using a graph requires taking the given output and looking on the graph to determine the corresponding input.

Example 9

Given the graph below,

a) Evaluate $f(2)$

b) Solve $f(x) = 4$



a) To evaluate $f(2)$, we find the input of $x=2$ on the horizontal axis. Moving up to the graph gives the point (2, 1), giving an output of $y=1$. So $f(2) = 1$

b) To solve $f(x) = 4$, we find the value 4 on the vertical axis because if $f(x) = 4$ then 4 is the output. Moving horizontally across the graph gives two points with the output of 4: (-1,4) and (3,4). These give the two solutions to $f(x) = 4$: $x = -1$ or $x = 3$

This means $f(-1)=4$ and $f(3)=4$, or when the input is -1 or 3, the output is 4.

Formulas as Functions

When possible, it is very convenient to define relationships using formulas. If it is possible to express the output as a formula involving the input quantity, then we can define a function.

Example 10

Express the relationship $2n + 6p = 12$ as a function $p = f(n)$ if possible.

To express the relationship in this form, we need to be able to write the relationship where p is a function of n, which means writing it as p = [something involving n].

$2n + 6p = 12$ subtract $2n$ from both sides

$6p = 12 - 2n$ divide both sides by 6 and simplify

$$p = \frac{12-2n}{6} = \frac{12}{6} - \frac{2n}{6} = 2 - \frac{1}{3}n$$

Having rewritten the formula as p=, we can now express p as a function:

$p = f(n) = 2 - \frac{1}{3}n$

It is important to note that not every relationship can be expressed as a function with a formula.

Note the important feature of an equation written as a function is that the output value can be determined directly from the input by doing evaluations - no further solving is required. This allows the relationship to act as a magic box that takes an input, processes it, and returns an output. Modern technology and computers rely on these functional relationships, since the evaluation of the function can be programmed into machines, whereas solving things is much more challenging.

Example 11

Express the relationship $x^2 + y^2 = 1$ as a function *y = f(x)* if possible.

If we try to solve for *y* in this equation:

$y^2 = 1 - x^2$

$y = \pm\sqrt{1 - x^2}$

We end up with two outputs corresponding to the same input, so this relationship cannot be represented as a single function *y = f(x)*

As with tables and graphs, it is common to evaluate and solve functions involving formulas. Evaluating will require replacing the input variable in the formula with the value provided and calculating. Solving will require replacing the output variable in the formula with the value provided, and solving for the input(s) that would produce that output.

Example 12

Given the function $k(t) = t^3 + 2$

a) Evaluate *k*(2)

b) Solve *k(t)* = 1

a) To evaluate *k*(2), we plug in the input value 2 into the formula wherever we see the input variable *t*, then simplify

$k(2) = 2^3 + 2$

$k(2) = 8 + 2$

So *k*(2) = 10

b) To solve *k(t)* = 1, we set the formula for *k(t)* equal to 1, and solve for the input value that will produce that output

k(t) = 1	substitute the original formula $k(t)=t^3+2$
$t^3+2=1$	subtract 2 from each side
$t^3=-1$	take the cube root of each side
$t=-1$	

When solving an equation using formulas, you can check your answer by using your solution in the original equation to see if your calculated answer is correct.

We want to know if $k(t)=1$ is true when $t=-1$.

$k(-1)=(-1)^3+2$

$= -1+2$

$=$ 1 which was the desired result.

Example 13

Given the function $h(p)=p^2+2p$

a) Evaluate *h*(4)

b) Solve *h(p)* = 3

To evaluate *h*(4) we substitute the value 4 for the input variable *p* in the given function.

a) $h(4)=(4)^2+2(4)$

= 16 + 8

= 24

b) *h(p)* = 3	Substitute the original function $h(p)=p^2+2p$
$p^2+2p=3$	This is quadratic, so we can rearrange the equation to get it = 0
$p^2+2p-3=0$	subtract 3 from each side
$p^2+2p-3=0$	this is factorable, so we factor it
$(p+3)(p-1)=0$	

By the zero factor theorem since $(p+3)(p-1)=0$, either $(p+3)=0$ or $(p-1)=0$ (or both of them equal 0) and so we solve both equations for *p*, finding *p* = -3 from the first equation and *p* = 1 from the second equation.

This gives us the solution: *h(p)* = 3 when *p* = 1 or *p* = -3

Basic Toolkit Functions

In this text, we will be exploring functions – the shapes of their graphs, their unique features, their equations, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of elements to build from. We call these our "toolkit of functions" – a set of basic named functions for which we know the graph, equation, and special features.

For these definitions we will use *x* as the input variable and *f(x)* as the output variable.

Toolkit Functions

Linear

Constant: $f(x) = c$, where *c* is a constant (number)

Identity: $f(x) = x$

Absolute Value: $f(x) = |x|$

Power

Quadratic: $f(x) = x^2$

Cubic: $f(x) = x^3$

Reciprocal: $f(x) = \frac{1}{x}$

Reciprocal squared: $f(x) = \frac{1}{x^2}$

Square root: $f(x) = \sqrt[2]{x} = \sqrt{x}$

Cube root: $f(x) = \sqrt[3]{x}$

You will see these toolkit functions, combinations of toolkit functions, their graphs and their transformations frequently throughout this book. In order to successfully follow along later in the book, it will be very helpful if you can recognize these toolkit functions and their features quickly by name, equation, graph and basic table values.

Not every important equation can be written as $y = f(x)$. An example of this is the equation of a circle. Recall the distance formula for the distance between two points:

$$dist = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

A circle with radius *r* with center at (*h*, *k*) can be described as all points (*x*, *y*) a distance of *r* from the center, so using the distance formula, $r = \sqrt{(x-h)^2 + (y-k)^2}$, giving

Equation of a circle

A circle with radius *r* with center (*h*, *k*) has equation $r^2 = (x-h)^2 + (y-k)^2$

Tutorial – 18

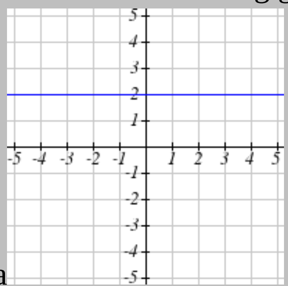
1. The amount of garbage, G, produced by a city with population p is given by $G = f(p)$. G is measured in tons per week, and p is measured in thousands of people.
   a. The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function f.
   b. Explain the meaning of the statement $f(5) = 2$.

2. The number of cubic yards of dirt, D, needed to cover a garden with area a square feet is given by $D = g(a)$.
   a. A garden with area 5000 ft^2 requires 50 cubic yards of dirt. Express this information in terms of the function g.
   b. Explain the meaning of the statement $g(100) = 1$.

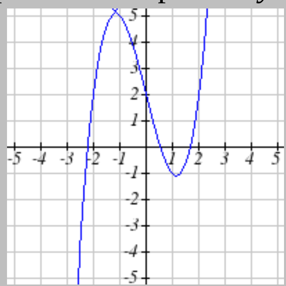
3. Let $f(t)$ be the number of ducks in a lake t years after 1990. Explain the meaning of each statement:
   a. $f(5) = 30$ b. $f(10) = 40$

4. Let $h(t)$ be the height above ground, in feet, of a rocket t seconds after launching. Explain the meaning of each statement:
   a. $h(1) = 200$ b. $h(2) = 350$

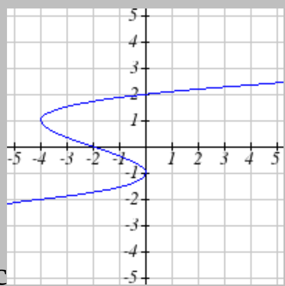
5. Select all of the following graphs which represent y as a function of x.



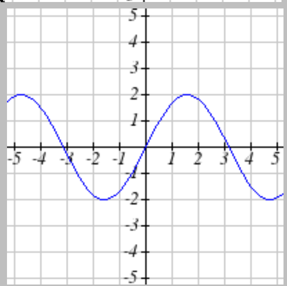
a



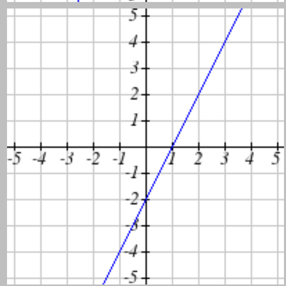
b



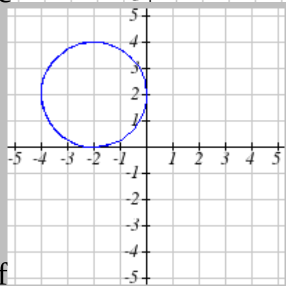
c



d

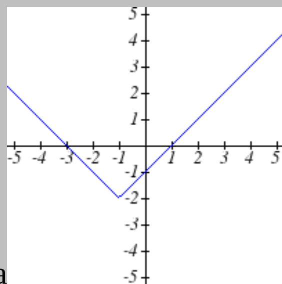


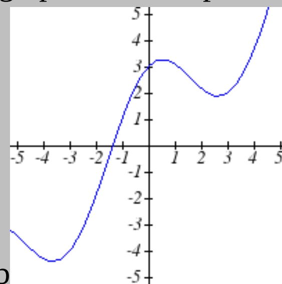
e

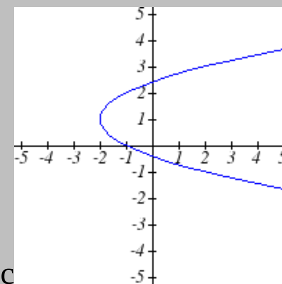


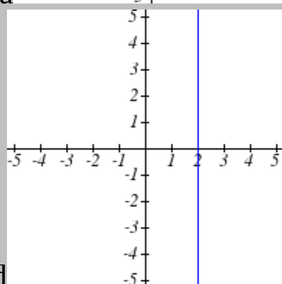
f

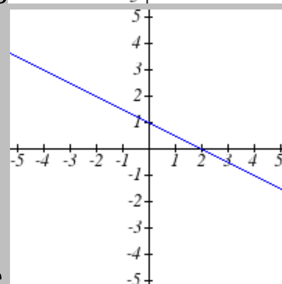
6. Select all of the following graphs which represent *y* as a function of *x*.

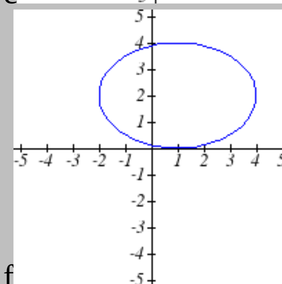
a


b


c


d


e


f


7. Select all of the following tables which represent *y* as a function of *x*.

a.

x	5	10	15
y	3	8	14

b.

x	5	10	15
y	3	8	8

c.

x	5	10	10
y	3	8	14

8. Select all of the following tables which represent *y* as a function of *x*.

a.

x	2	6	13
y	3	10	10

b.

x	2	6	6
y	3	10	14

c.

x	2	6	13
y	3	10	14

9. Select all of the following tables which represent *y* as a function of *x*.

a.

x	***y***
0	-2
3	1
4	6
8	9
3	1

b.

x	***y***
-1	-4
2	3
5	4
8	7
12	11

c.

x	***y***
0	-5
3	1
3	4
9	8
16	13

d.

x	***y***
-1	-4
1	2
4	2
9	7
12	13

10. Select all of the following tables which represent *y* as a function of *x*.

a.

x	***y***
-4	-2
3	2
6	4
9	7
12	16

b.

x	***y***
-5	-3
2	1
2	4
7	9
11	10

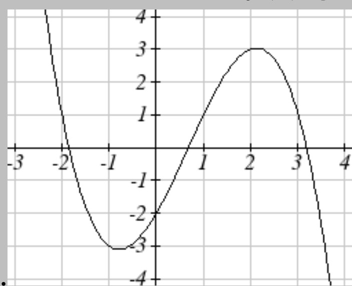
c.

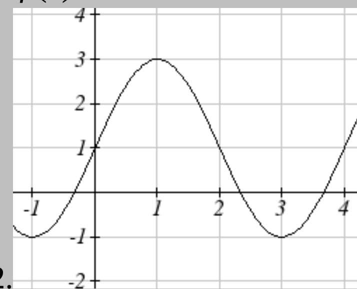
x	***y***
-1	-3
1	2
5	4
9	8
1	2

d.

x	***y***
-1	-5
3	1
5	1
8	7
14	12

Given each function $f(x)$ graphed, evaluate $f(1)$ and $f(3)$

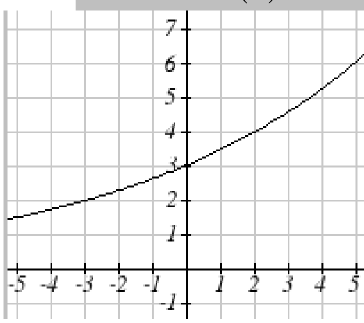
11. 

12. 

13. Given the function $g(x)$ graphed here,

a. Evaluate $g(2)$

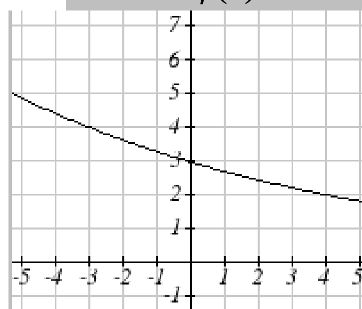
b. Solve $g(x) = 2$



14. Given the function $f(x)$ graphed here.

c. Evaluate $f(4)$

d. Solve $f(x)=4$



15. Based on the table below,

a. Evaluate $f(3)$ b. Solve $f(x)=1$

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	74	28	1	53	56	3	36	45	14	47

16. Based on the table below,

a. Evaluate $f(8)$ b. Solve $f(x)=7$

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	62	8	7	38	86	73	70	39	75	34

For each of the following functions, evaluate: $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, and $f(2)$

17. $f(x) = 4 - 2x$

18. $f(x) = 8 - 3x$

19. $f(x) = 8x^2 - 7x + 3$

20. $f(x) = 6x^2 - 7x + 4$

21. $f(x) = -x^3 + 2x$

22. $f(x) = 5x^4 + x^2$

23. $f(x) = 3 + \sqrt{x+3}$

24. $f(x) = 4 - \sqrt[3]{x-2}$

25. $f(x) = (x-2)(x+3)$

26. $f(x) = (x+3)(x-1)^2$

27. $f(x) = \dfrac{x-3}{x+1}$

28. $f(x) = \dfrac{x-2}{x+2}$

29. $f(x) = 2^x$

30. $f(x) = 3^x$

31. Suppose $f(x) = x^2 + 8x - 4$. Compute the following:

a. $f(-1) + f(1)$ b. $f(-1) - f(1)$

32. Suppose $f(x) = x^2 + x + 3$. Compute the following:

a. $f(-2) + f(4)$ b. $f(-2) - f(4)$

33. Let $f(t) = 3t + 5$

a. Evaluate $f(0)$ b. Solve $f(t) = 0$

34.Let $g(p) = 6 - 2p$

a. Evaluate $g(0)$ b. Solve $g(p) = 0$

35. Match each function name with its equation.

a. $y = x$

b. $y = x^3$

c. $y = \sqrt[3]{x}$

d. $y = \frac{1}{x}$

e. $y = x^2$

f. $y = \sqrt{x}$

g. $y = |x|$

h. $y = \frac{1}{x^2}$

i. Cube root
ii. Reciprocal
iii. Linear
iv. Square Root
v. Absolute Value
vi. Quadratic
vii. Reciprocal Squared
viii. Cubic

****

9.2 Domain and Range

One of our main goals in mathematics is to model the real world with mathematical functions. In doing so, it is important to keep in mind the limitations of those models we create.

This table shows a relationship between circumference and height of a tree as it grows.

Circumference, *c*	1.7	2.5	5.5	8.2	13.7
Height, *h*	24.5	31	45.2	54.6	92.1

While there is a strong relationship between the two, it would certainly be ridiculous to talk about a tree with a circumference of -3 feet, or a height of 3000 feet. When we identify limitations on the inputs and outputs of a function, we are determining the domain and range of the function.

Domain and Range

Domain: The set of possible input values to a function
Range: The set of possible output values of a function

Example 1

Using the tree table above, determine a reasonable domain and range.

We could combine the data provided with our own experiences and reason to approximate the domain and range of the function $h = f(c)$. For the domain, possible values for the input circumference c, it doesn't make sense to have negative values, so $c > 0$. We could make an educated guess at a maximum reasonable value, or look up that the maximum circumference measured is about 119 feet[1]. With this information we would say a reasonable domain is $0 < c \leq 119$ feet.

Similarly for the range, it doesn't make sense to have negative heights, and the maximum height of a tree could be looked up to be 379 feet, so a reasonable range is $0 < h \leq 379$ feet.

Example 2

When sending a letter through the United States Postal Service, the price depends upon the weight of the letter[2], as shown in the table below. Determine the domain and range.

Letters	
Weight not Over	Price
1 ounce	$0.44
2 ounces	$0.61
3 ounces	$0.78
3.5 ounces	$0.95

Suppose we notate Weight by w and Price by p, and set up a function named P, where Price, p is a function of Weight, w. $p = P(w)$.

Since acceptable weights are 3.5 ounces or less, and negative weights don't make sense, the domain would be $0 < w \leq 3.5$. Technically 0 could be included in the domain, but logically it would mean we are mailing nothing, so it doesn't hurt to leave it out.

[1] http://en.wikipedia.org/wiki/Tree, retrieved July 19, 2010
[2] http://www.usps.com/prices/first-class-mail-prices.htm, retrieved July 19, 2010

Since possible prices are from a limited set of values, we can only define the range of this function by listing the possible values. The range is *p* = \$0.44, \$0.61, \$0.78, or \$0.95.

Try it Now

1. The population of a small town in the year 1960 was 100 people. Since then the population has grown to 1400 people reported during the 2010 census. Choose descriptive variables for your input and output and use interval notation to write the domain and range.

Notation

In the previous examples, we used inequalities to describe the domain and range of the functions. This is one way to describe intervals of input and output values, but is not the only way. Let us take a moment to discuss notation for domain and range.

Using inequalities, such as $0 < c \leq 163$, $0 < w \leq 3.5$, and $0 < h \leq 379$ imply that we are interested in all values between the low and high values, including the high values in these examples.

However, occasionally we are interested in a specific list of numbers like the range for the price to send letters, *p* = \$0.44, \$0.61, \$0.78, or \$0.95. These numbers represent a set of specific values: {0.44, 0.61, 0.78, 0.95}

Representing values as a set, or giving instructions on how a set is built, leads us to another type of notation to describe the domain and range.

Suppose we want to describe the values for a variable *x* that are 10 or greater, but less than 30. In inequalities, we would write $10 \leq x < 30$.

When describing domains and ranges, we sometimes extend this into **set-builder notation**, which would look like this: $\{x \mid 10 \leq x < 30\}$. The curly brackets {} are read as "the set of", and the vertical bar | is read as "such that", so altogether we would read $\{x \mid 10 \leq x < 30\}$ as "the set of *x*-values such that 10 is less than or equal to *x* and *x* is less than 30."

When describing ranges in set-builder notation, we could similarly write something like $\{f(x) \mid 0 < f(x) < 100\}$, or if the output had its own variable, we could use it. So for our tree height example above, we could write for the range $\{h \mid 0 < h \leq 379\}$. In set-builder notation, if a domain or range is not limited, we could write $\{t \mid t \text{ is a real number}\}$, or $\{t \mid t \in \mathbb{R}\}$, read as "the set of *t*-values such that *t* is an element of the set of real numbers.

A more compact alternative to set-builder notation is **interval notation**, in which intervals of values are referred to by the starting and ending values. Curved parentheses are used for “strictly less than,” and square brackets are used for “less than or equal to.” Since infinity is not a number, we can’t include it in the interval, so we always use curved parentheses with ∞ and -∞. The table below will help you see how inequalities correspond to set-builder notation and interval notation:

Inequality	**Set Builder Notation**	**Interval notation**
$5 < h \leq 10$	$\{h \mid 5 < h \leq 10\}$	(5, 10]
$5 \leq h < 10$	$\{h \mid 5 \leq h < 10\}$	[5, 10)
$5 < h < 10$	$\{h \mid 5 < h < 10\}$	(5, 10)
$h < 10$	$\{h \mid h < 10\}$	$(-\infty, 10)$
$h \geq 10$	$\{h \mid h \geq 10\}$	$[10, \infty)$
all real numbers	$\{h \mid h \in \mathbb{R}\}$	$(-\infty, \infty)$

To combine two intervals together, using inequalities or set-builder notation we can use the word “or”. In interval notation, we use the union symbol, $\cup$, to combine two unconnected intervals together.

Example 3

Describe the intervals of values shown on the line graph below using set builder and interval notations.

To describe the values, *x*, that lie in the intervals shown above we would say, “*x* is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5.”

As an inequality it is: $1 \leq x \leq 3$ or $x > 5$

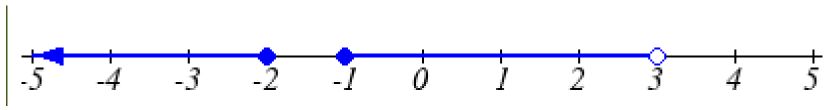
In set builder notation: $\{x \mid 1 \leq x \leq 3 \text{ or } x > 5\}$

In interval notation: $[1, 3] \cup (5, \infty)$

Remember when writing or reading interval notation:
Using a square bracket [means the start value is included in the set
Using a parenthesis (means the start value is not included in the set

Try it Now

2. Given the following interval, write its meaning in words, set builder notation, and interval notation.



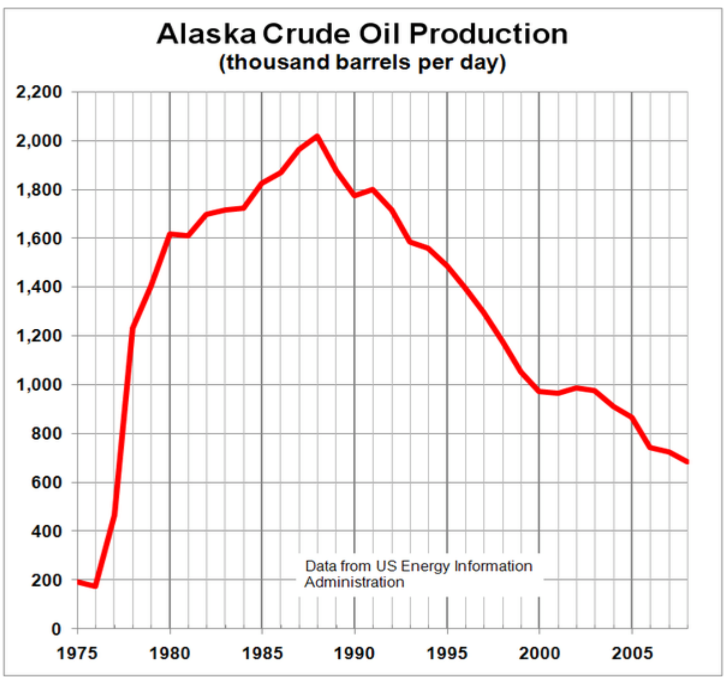
Domain and Range from Graphs

We can also talk about domain and range based on graphs. Since domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the graph. Remember that input values are almost always shown along the horizontal axis of the graph. Likewise, since range is the set of possible output values, the range of a graph we can see from the possible values along the vertical axis of the graph.

Be careful – if the graph continues beyond the window on which we can see the graph, the domain and range might be larger than the values we can see.

Example 4

Determine the domain and range of the graph below.



In the graph above[3], the input quantity along the horizontal axis appears to be "year", which we could notate with the variable *y*. The output is "thousands of barrels of oil per day", which we might notate with the variable *b*, for barrels. The graph would likely continue to the left and right beyond what is shown, but based on the portion of the graph that is shown to us, we can determine the domain is $1975 \le y \le 2008$, and the range is approximately $180 \le b \le 2010$.

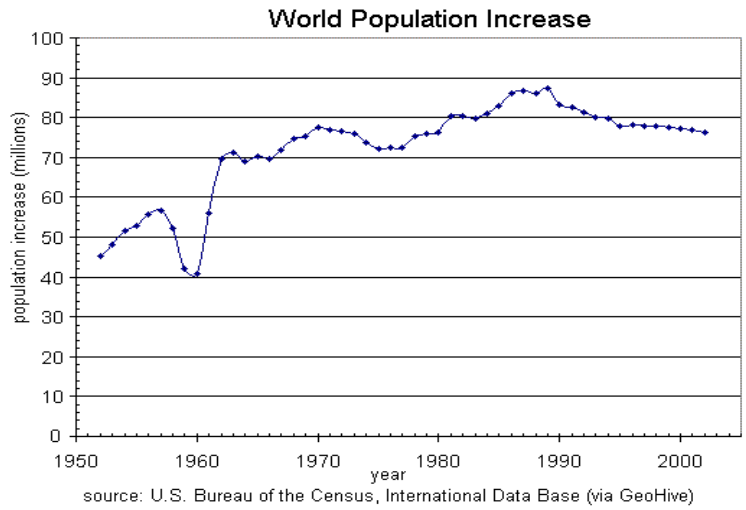
[3] http://commons.wikimedia.org/wiki/File:Alaska_Crude_Oil_Production.PNG, CC-BY-SA, July 19, 2010

In interval notation, the domain would be [1975, 2008] and the range would be about [180, 2010]. For the range, we have to approximate the smallest and largest outputs since they don't fall exactly on the grid lines.

Remember that, as in the previous example, x and y are not always the input and output variables. Using descriptive variables is an important tool to remembering the context of the problem.

Try it Now

3. Given the graph below write the domain and range in interval notation



Domains and Ranges of the Toolkit functions

We will now return to our set of toolkit functions to note the domain and range of each.

Constant Function: $f(x) = c$

The domain here is not restricted; x can be anything. When this is the case we say the domain is all real numbers. The outputs are limited to the constant value of the function.

Domain: $(-\infty, \infty)$

Range: $[c]$

Since there is only one output value, we list it by itself in square brackets.

Identity Function: $f(x) = x$

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

Quadratic Function: $f(x) = x^2$

Domain: $(-\infty, \infty)$

Range: $[0, \infty)$

Multiplying a negative or positive number by itself can only yield a positive output.

<u>Cubic Function</u>: $f(x) = x^3$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

<u>Reciprocal</u>: $f(x) = \dfrac{1}{x}$
Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$
We cannot divide by 0 so we must exclude 0 from the domain.
One divide by any value can never be 0, so the range will not include 0.

<u>Reciprocal squared</u>: $f(x) = \dfrac{1}{x^2}$
Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(0, \infty)$
We cannot divide by 0 so we must exclude 0 from the domain.

<u>Cube Root</u>: $f(x) = \sqrt[3]{x}$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

<u>Square Root</u>: $f(x) = \sqrt[2]{x}$, commonly just written as, $f(x) = \sqrt{x}$
Domain: $[0, \infty)$
Range: $[0, \infty)$
When dealing with the set of real numbers we cannot take the square root of a negative number so the domain is limited to 0 or greater.

<u>Absolute Value Function</u>: $f(x) = |x|$
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$
Since absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.

Example 4.5

Find the domain of each function: a) $f(x) = 2\sqrt{x+4}$ b) $g(x) = \dfrac{3}{6-3x}$

a) Since we cannot take the square root of a negative number, we need the inside of the square root to be non-negative.
$x+4\geq 0$ when $x\geq -4$.
The domain of *f(x)* is $[-4,\infty)$.

b) We cannot divide by zero, so we need the denominator to be non-zero.
$6-3x=0$ when x = 2, so we must exclude 2 from the domain.
The domain of *g(x)* is $(-\infty,2)\cup(2,\infty)$.

Piecewise Functions

In the toolkit functions we introduced the absolute value function $f(x)=|x|$.

With a domain of all real numbers and a range of values greater than or equal to 0, the absolute value can be defined as the magnitude or modulus of a number, a real number value regardless of sign, the size of the number, or the distance from 0 on the number line. All of these definitions require the output to be greater than or equal to 0.

If we input 0, or a positive value the output is unchanged
$f(x)=x \quad \text{if} \quad x\geq 0$

If we input a negative value the sign must change from negative to positive.
$f(x)=-x \quad \text{if} \quad x<0$ since multiplying a negative value by -1 makes it positive.

Since this requires two different processes or pieces, the absolute value function is often called the most basic piecewise defined function.

Piecewise Function

A **piecewise function** is a function in which the formula used depends upon the domain the input lies in. We notate this idea like:

$$f(x)=\begin{cases}\text{formula 1} & \text{if} \quad \text{domain to use formula 1}\\ \text{formula 2} & \text{if} \quad \text{domain to use formula 2}\\ \text{formula 3} & \text{if} \quad \text{domain to use formula 3}\end{cases}$$

Example 5

A museum charges \$5 per person for a guided tour with a group of 1 to 9 people, or a fixed \$50 fee for 10 or more people in the group. Set up a function relating the number of people, *n*, to the cost, *C*.

To set up this function, two different formulas would be needed. C = 5n would work for n values under 10, and C = 50 would work for values of n ten or greater. Notating this:

$$C(n)=\begin{cases}5n & if \quad 0<n<10\\ 50 & if \quad n\geq 10\end{cases}$$

Example 6

A cell phone company uses the function below to determine the cost, *C*, in dollars for *g* gigabytes of data transfer.

$$C(g)=\begin{cases}25 & if \quad 0<g<2\\ 25+10(g-2) & if \quad g\ge 2\end{cases}$$

Find the cost of using 1.5 gigabytes of data, and the cost of using 4 gigabytes of data.

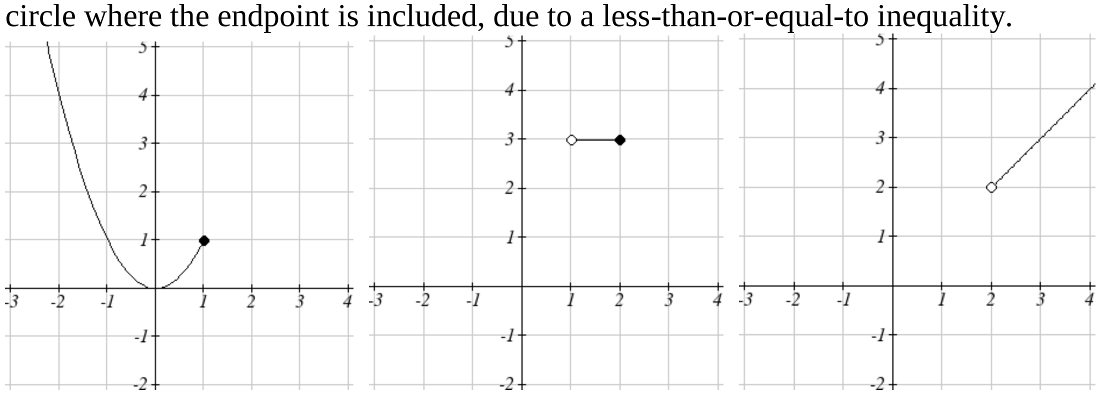
To find the cost of using 1.5 gigabytes of data, *C*(1.5), we first look to see which piece of domain our input falls in. Since 1.5 is less than 2, we use the first formula, giving *C*(1.5) = $25.

The find the cost of using 4 gigabytes of data, *C*(4), we see that our input of 4 is greater than 2, so we'll use the second formula. *C*(4) = 25 + 10(4-2) = $45.

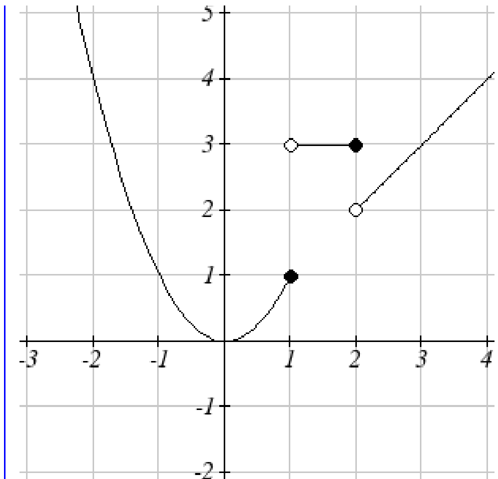
Example 7

Sketch a graph of the function $f(x)=\begin{cases}x^2 & if \quad x\le 1\\ 3 & if \quad 1<x\le 2\\ x & if \quad x>2\end{cases}$

Since each of the component functions are from our library of Toolkit functions, we know their shapes. We can imagine graphing each function, then limiting the graph to the indicated domain. At the endpoints of the domain, we put open circles to indicate where the endpoint is not included, due to a strictly-less-than inequality, and a closed circle where the endpoint is included, due to a less-than-or-equal-to inequality.



Now that we have each piece individually, we combine them onto the same graph:



Try it Now

4. At Pierce College during the 2009-2010 school year tuition rates for in-state residents were \$89.50 per credit for the first 10 credits, \$33 per credit for credits 11-18, and for over 18 credits the rate is \$73 per credit[4]. Write a piecewise defined function for the total tuition, *T*, at Pierce College during 2009-2010 as a function of the number of credits taken, *c*. Be sure to consider a reasonable domain and range.

Try it Now Answers

1. Domain; *y* = years [1960,2010] ; Range, *p* = population, [100,1400]

2. a. Values that are less than or equal to -2, or values that are greater than or equal to -1 and less than 3
   b. $\{x \mid x \le -2 \text{ or } -1 \le x < 3\}$
   c. $(-\infty,-2] \cup [-1,3)$

3. Domain; *y*=years, [1952,2002] ; Range, *p*=population in millions, [40,88]

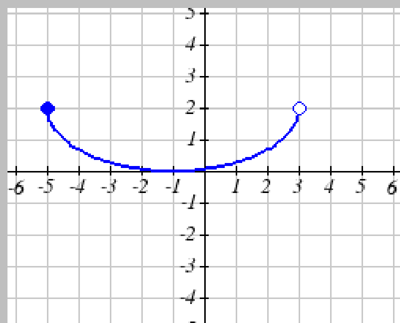
4. $T(c) = \begin{cases} 89.5c & \text{if} \quad c \le 10 \\ 895 + 33(c-10) & \text{if} \quad 10 < c \le 18 \\ 1159 + 73(c-18) & \text{if} \quad c > 18 \end{cases}$ Tuition, *T*, as a function of credits, *c*.

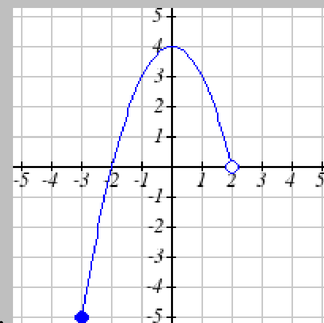
Reasonable domain should be whole numbers 0 to (answers may vary), e.g. $[0,23]$
Reasonable range should be \$0 – (answers may vary), e.g. $[0,1524]$

[4] https://www.pierce.ctc.edu/dist/tuition/ref/files/0910_tuition_rate.pdf, retrieved August 6, 2010

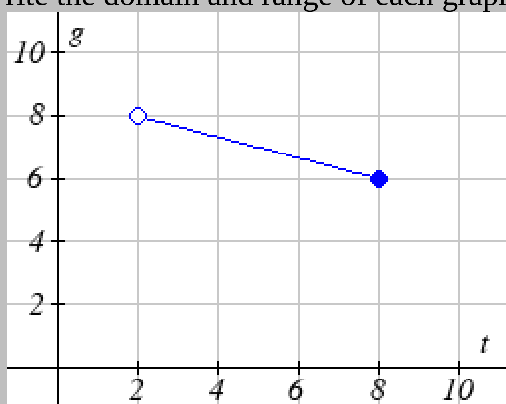
Tutorial – 19

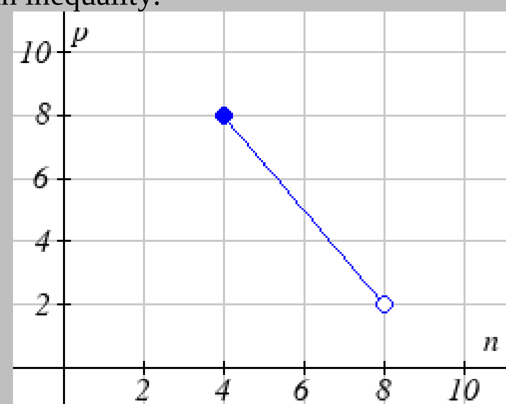
Write the domain and range of the function using interval notation.

1. 

2. 

Write the domain and range of each graph as an inequality.

3. 

4. 

Find the domain of each function

5. $f(x)=3\sqrt{x-2}$

6. $f(x)=5\sqrt{x+3}$

7. $f(x)=3-\sqrt{6-2x}$

8. $f(x)=5-\sqrt{10-2x}$

9. $f(x)=\dfrac{9}{x-6}$

10. $f(x)=\dfrac{6}{x-8}$

11. $f(x)=\dfrac{3x+1}{4x+2}$

12. $f(x)=\dfrac{5x+3}{4x-1}$

13. $f(x)=\dfrac{\sqrt{x+4}}{x-4}$

14. $f(x)=\dfrac{\sqrt{x+5}}{x-6}$

15. $f(x)=\dfrac{x-3}{x^2+9x-22}$

16. $f(x)=\dfrac{x-8}{x^2+8x-9}$

Given each function, evaluate: $f(-1)$, $f(0)$, $f(2)$, $f(4)$

17. $f(x)=\begin{cases} 7x+3 & \text{if} \quad x<0 \\ 7x+6 & \text{if} \quad x\geq 0 \end{cases}$

18. $f(x)=\begin{cases} 4x-9 & \text{if} \quad x<0 \\ 4x-18 & \text{if} \quad x\geq 0 \end{cases}$

19. $f(x)=\begin{cases} x^2-2 & \text{if} \quad x<2 \\ 4+|x-5| & \text{if} \quad x\geq 2 \end{cases}$

20. $f(x)=\begin{cases} 4-x^3 & \text{if} \quad x<1 \\ \sqrt{x+1} & \text{if} \quad x\geq 1 \end{cases}$

21. $f(x)=\begin{cases} 5x & \text{if} \quad x<0 \\ 3 & \text{if} \quad 0\leq x\leq 3 \\ x^2 & \text{if} \quad x>3 \end{cases}$

22. $f(x)=\begin{cases} x^3+1 & \text{if} \quad x<0 \\ 4 & \text{if} \quad 0\leq x\leq 3 \\ 3x+1 & \text{if} \quad x>3 \end{cases}$

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9.3 Composition of Functions

Suppose we wanted to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The temperature depends on the day, and the cost depends on the temperature. Using descriptive variables, we can notate these two functions.

The first function, *C(T)*, gives the cost *C* of heating a house when the average daily temperature is *T* degrees Celsius, and the second, *T(d)*, gives the average daily temperature of a particular city on day *d* of the year. If we wanted to determine the cost of heating the house on the 5^{th} day of the year, we could do this by linking our two functions together, an idea called composition of functions. Using the function *T(d)*, we could evaluate *T*(5) to determine the average daily temperature on the 5^{th} day of the year. We could then use that temperature as the input to the *C(T)* function to find the cost to heat the house on the 5^{th} day of the year: *C(T(5)).*

Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a **composition of functions**. We write *f(g(x))*, and read this as "*f* of *g* of *x*" or "*f* composed with *g* at *x*".

An alternate notation for composition uses the composition operator: $\circ$
$(f\circ g)(x)$ is read "*f* of *g* of *x*" or "*f* composed with *g* at *x*", just like *f(g(x)).*

Example 1

Suppose *c(s)* gives the number of calories burned doing *s* sit-ups, and *s(t)* gives the number of sit-ups a person can do in *t* minutes. Interpret *c*(*s*(3)).

When we are asked to interpret, we are being asked to explain the meaning of the expression in words. The inside expression in the composition is *s*(3). Since the input to the *s* function is time, the 3 is representing 3 minutes, and *s*(3) is the number of sit-ups that can be done in 3 minutes. Taking this output and using it as the input to the *c(s)* function will gives us the calories that can be burned by the number of sit-ups that can be done in 3 minutes.

Note that it is not important that the same variable be used for the output of the inside function and the input to the outside function. However, it *is* essential that the units on the output of the inside function match the units on the input to the outside function, if the units are specified.

Example 2

Suppose *f(x)* gives miles that can be driven in *x* hours, and *g(y)* gives the gallons of gas used in driving *y* miles. Which of these expressions is meaningful: *f(g(y))* or *g(f(x))*?

The expression *g(y)* takes miles as the input and outputs a number of gallons. The function *f(x)* is expecting a number of hours as the input; trying to give it a number of gallons as input does not make sense. Remember the units have to match, and number of gallons does not match number of hours, so the expression *f(g(y))* is meaningless.

The expression *f(x)* takes hours as input and outputs a number of miles driven. The function *g(y)* is expecting a number of miles as the input, so giving the output of the *f(x)* function (miles driven) as an input value for g(*y*), where gallons of gas depends on miles driven, does make sense. The expression *g(f(x))* makes sense, and will give the number of gallons of gas used, *g*, driving a certain number of miles, *f(x)*, in *x* hours.

Try it Now

1. In a department store you see a sign that says 50% off of clearance merchandise, so final cost *C* depends on the clearance price, *p*, according to the function *C(p)*. Clearance price, *p*, depends on the original discount, *d*, given to the clearance item, *p(d)*. Interpret *C(p(d))*.

Composition of Functions using Tables and Graphs

When working with functions given as tables and graphs, we can look up values for the functions using a provided table or graph, as discussed in section 1.1. We start evaluation from the provided input, and first evaluate the inside function. We can then use the output of the inside function as the input to the outside function. To remember this, always work from the inside out.

Example 3

Using the tables below, evaluate $f(g(3))$ and $g(f(4))$

x	*f(x)*
1	6
2	8
3	3
4	1

x	*g(x)*
1	3
2	5
3	2
4	7

To evaluate $f(g(3))$, we start from the inside with the value 3. We then evaluate the inside expression $g(3)$ using the table that defines the function *g*: $g(3) = 2$. We can then use that result as the input to the *f* function, so $g(3)$ is replaced by the equivalent value 2 and we get $f(2)$. Then using the table that defines the function *f*, we find that $f(2) = 8$. $f(g(3)) = f(2) = 8$.

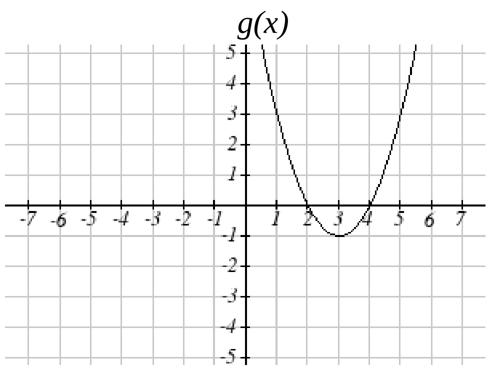
To evaluate $g(f(4))$, we first evaluate the inside expression $f(4)$ using the first table: $f(4) = 1$. Then using the table for *g* we can evaluate:
$g(f(4)) = g(1) = 3$

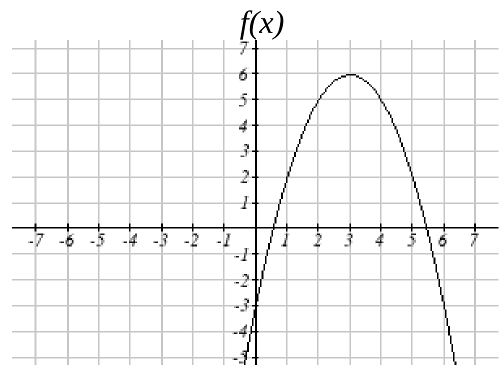
Try it Now

2. Using the tables from the example above, evaluate $f(g(1))$ and $g(f(3))$.

Example 4

Using the graphs below, evaluate $f(g(1))$.





To evaluate $f(g(1))$, we again start with the inside evaluation. We evaluate $g(1)$ using the graph of the *g(x)* function, finding the input of 1 on the horizontal axis and finding the output value of the graph at that input. Here, $g(1) = 3$. Using this value as the input to the *f* function, $f(g(1)) = f(3)$. We can then evaluate this by looking to the graph of the *f(x)* function, finding the input of 3 on the horizontal axis, and reading the output value of the graph at this input. Here, $f(3)=6$, so $f(g(1))=6$.

Try it Now

3. Using the graphs from the previous example, evaluate $g(f(2))$.

Composition using Formulas

When evaluating a composition of functions where we have either created or been given formulas, the concept of working from the inside out remains the same. First we evaluate the inside function using the input value provided, then use the resulting output as the input to the outside function.

Example 5

Given $f(t)=t^2-t$ and $h(x)=3x+2$, evaluate $f(h(1))$.

Since the inside evaluation is $h(1)$ we start by evaluating the *h(x)* function at 1:

$h(1)=3(1)+2=5$

Then $f(h(1))=f(5)$, so we evaluate the *f(t)* function at an input of 5:

$f(h(1))=f(5)=5^2-5=20$

Try it Now

4. Using the functions from the example above, evaluate $h(f(-2))$.

While we can compose the functions as above for each individual input value, sometimes it would be really helpful to find a single formula which will calculate the result of a composition *f(g(x))*. To do this, we will extend our idea of function evaluation. Recall that when we evaluate a function like $f(t)=t^2-t$, we put whatever value is inside the parentheses after the function name into the formula wherever we see the input variable.

Example 6

Given $f(t)=t^2-t$, evaluate $f(3)$ and $f(-2)$.

$f(3)=3^2-3$

$f(-2)=(-2)^2-(-2)$

We could simplify the results above if we wanted to

$f(3) = 3^2 - 3 = 9 - 3 = 6$
$f(-2) = (-2)^2 - (-2) = 4 + 2 = 6$

We are not limited, however, to using a numerical value as the input to the function. We can put anything into the function: a value, a different variable, or even an algebraic expression, provided we use the input expression everywhere we see the input variable.

Example 7

Using the function from the previous example, evaluate *f(a)*

This means that the input value for *t* is some unknown quantity *a*. As before, we evaluate by replacing the input variable *t* with the input quantity, in this case *a*.

$f(a) = a^2 - a$

The same idea can then be applied to expressions more complicated than a single letter.

Example 8

Using the same *f(t)* function from above, evaluate $f(x+2)$.

Everywhere in the formula for *f* where there was a *t*, we would replace it with the input $(x+2)$. Since in the original formula the input *t* was squared in the first term, the entire input $x+2$ needs to be squared when we substitute, so we need to use grouping parentheses. To avoid problems, it is advisable to always use parentheses around inputs.

$f(x+2) = (x+2)^2 - (x+2)$

We could simplify this expression further to $f(x+2) = x^2 + 3x + 2$ if we wanted to:

$f(x+2) = (x+2)(x+2) - (x+2)$ Use the "FOIL" technique (first, outside, inside, last)
$f(x+2) = x^2 + 2x + 2x + 4 - (x+2)$ distribute the negative sign
$f(x+2) = x^2 + 2x + 2x + 4 - x - 2$ combine like terms
$f(x+2) = x^2 + 3x + 2$

Example 9

Using the same function, evaluate $f(t^3)$.

Note that in this example, the same variable is used in the input expression and as the input variable of the function. This doesn't matter – we still replace the original input *t* in the formula with the new input expression, t^3.

$f(t^3) = (t^3)^2 - (t^3) = t^6 - t^3$

Try it Now

5. Given $g(x) = 3x - \sqrt{x}$, evaluate $g(t-2)$.

This now allows us to find an expression for a composition of functions. If we want to find a formula for *f(g(x))*, we can start by writing out the formula for *g(x)*. We can then evaluate the function *f(x)* at that expression, as in the examples above.

Example 10

Let $f(x) = x^2$ and $g(x) = \dfrac{1}{x} - 2x$, find *f(g(x))* and *g(f(x))*.

To find *f(g(x))*, we start by evaluating the inside, writing out the formula for *g(x)*

$$g(x) = \frac{1}{x} - 2x$$

We then use the expression $\left(\dfrac{1}{x} - 2x\right)$ as input for the function *f*.

$$f(g(x)) = f\left(\frac{1}{x} - 2x\right)$$

We then evaluate the function *f(x)* using the formula for *g(x)* as the input.

Since $f(x) = x^2$ then $f\left(\dfrac{1}{x} - 2x\right) = \left(\dfrac{1}{x} - 2x\right)^2$

This gives us the formula for the composition: $f(g(x)) = \left(\dfrac{1}{x} - 2x\right)^2$

Likewise, to find *g(f(x))*, we evaluate the inside, writing out the formula for *f(x)*

$$g(f(x)) = g\left(x^2\right)$$

Now we evaluate the function *g(x)* using x^2 as the input.

$$g(f(x)) = \frac{1}{x^2} - 2x^2$$

Try it Now

6. Let $f(x) = x^3 + 3x$ and $g(x) = \sqrt{x}$, find *f(g(x))* and *g(f(x))*.

Example 11

A city manager determines that the tax revenue, *R*, in millions of dollars collected on a population of *p* thousand people is given by the formula $R(p) = 0.03p + \sqrt{p}$, and that the city's population, in thousands, is predicted to follow the formula $p(t) = 60 + 2t + 0.3t^2$, where *t* is measured in years after 2010. Find a formula for the tax revenue as a function of the year.

Since we want tax revenue as a function of the year, we want year to be our initial input, and revenue to be our final output. To find revenue, we will first have to predict the city population, and then use that result as the input to the tax function. So we need to find *R(p(t))*. Evaluating this,

$$R(p(t)) = R\left(60 + 2t + 0.3t^2\right) = 0.03\left(60 + 2t + 0.3t^2\right) + \sqrt{60 + 2t + 0.3t^2}$$

This composition gives us a single formula which can be used to predict the tax revenue during a given year, without needing to find the intermediary population value.

For example, to predict the tax revenue in 2017, when $t = 7$ (because *t* is measured in years after 2010)

$$R(p(7)) = 0.03\left(60 + 2(7) + 0.3(7)^2\right) + \sqrt{60 + 2(7) + 0.3(7)^2} \approx 12.079 \text{ million dollars}$$

In some cases, it is desirable to decompose a function – to write it as a composition of two simpler functions.

Example 12

Write $f(x) = 3 + \sqrt{5 - x^2}$ as the composition of two functions.

We are looking for two functions, *g* and *h*, so $f(x) = g(h(x))$. To do this, we look for a function inside a function in the formula for *f(x)*. As one possibility, we might notice that $5 - x^2$ is the inside of the square root. We could then decompose the function as:

$h(x) = 5 - x^2$

$g(x) = 3 + \sqrt{x}$

We can check our answer by recomposing the functions:

$g(h(x)) = g\left(5 - x^2\right) = 3 + \sqrt{5 - x^2}$

Note that this is not the only solution to the problem. Another non-trivial decomposition would be $h(x) = x^2$ and $g(x) = 3 + \sqrt{5 - x}$

Try it Now Answers

1. The final cost, C, depends on the clearance price, p, which is based on the original discount, d. (Or the original discount d, determines the clearance price and the final cost is half of the clearance price.)
2. $f(g(1)) = f(3) = 3$ and $g(f(3)) = g(3) = 2$
3. $g(f(2)) = g(5) = 3$
4. $h(f(-2)) = h(6) = 20$ *did you remember to insert your input values using parentheses?*
5. $g(t-2) = 3(t-2) - \sqrt{(t-2)}$
6. $f(g(x)) = f\left(\sqrt{x}\right) = \left(\sqrt{x}\right)^3 + 3\left(\sqrt{x}\right)$

   $g(f(x)) = g\left(x^3 + 3x\right) = \sqrt{\left(x^3 + 3x\right)}$

Tutorial – 20

Given each pair of functions, calculate $f\left(g(0)\right)$ and $g\left(f(0)\right)$.

1. $f(x) = 4x + 8,\ g(x) = 7 - x^2$
2. $f(x) = 5x + 7,\ g(x) = 4 - 2x^2$
3. $f(x) = \sqrt{x+4},\ g(x) = 12 - x^3$
4. $f(x) = \dfrac{1}{x+2},\ g(x) = 4x + 3$

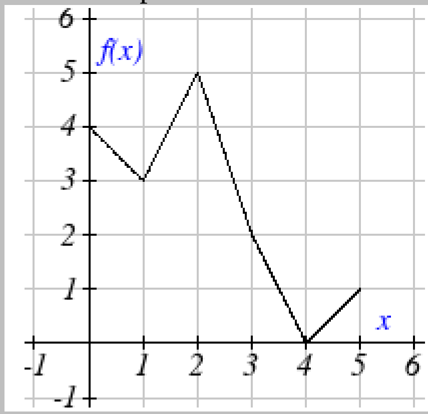
Use the table of values to evaluate each expression

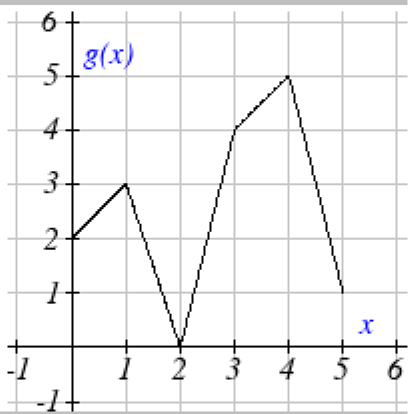
5. $f(g(8))$
6. $f\left(g(5)\right)$
7. $g(f(5))$
8. $g\left(f(3)\right)$
9. $f(f(4))$
10. $f\left(f(1)\right)$
11. $g(g(2))$
12. $g\left(g(6)\right)$

x	$f(x)$	$g(x)$
0	7	9
1	6	5
2	5	6
3	8	2
4	4	1
5	0	8
6	2	7
7	1	3
8	9	4
9	3	0

Use the graphs to evaluate the expressions below.

13. $f(g(3))$
14. $f(g(1))$
15. $g(f(1))$
16. $g(f(0))$
17. $f(f(5))$
18. $f(f(4))$
19. $g(g(2))$
20. $g(g(0))$





For each pair of functions, find $f(g(x))$ and $g(f(x))$. Simplify your answers.

21. $f(x)=\frac{1}{x-6}$, $g(x)=\frac{7}{x}+6$
22. $f(x)=\frac{1}{x-4}$, $g(x)=\frac{2}{x}+4$
23. $f(x)=x^2+1$, $g(x)=\sqrt{x+2}$
24. $f(x)=\sqrt{x}+2$, $g(x)=x^2+3$
25. $f(x)=|x|$, $g(x)=5x+1$
26. $f(x)=\sqrt[3]{x}$, $g(x)=\frac{x+1}{x^3}$

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CHAPTER 10 – TRIGONOMETRY

Learning Objectives:

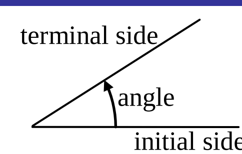
- ✓ *Understand the definition of the different types of angles and measure them in degrees and radians.*
- ✓ *Demonstrate an understanding of trigonometric identities.*
- ✓ *Use the law of sines and cosines to solve a triangle and real life problems*

10.1 Angles

Because many applications involving circles also involve a rotation of the circle, it is natural to introduce a measure for the rotation, or angle, between two rays (line segments) emanating from the center of a circle. The angle measurement you are most likely familiar with is degrees, so we'll begin there.

Measure of an Angle

The **measure of an angle** is a measurement between two intersecting lines, line segments or rays, starting at the **initial side** and ending at the **terminal side**. It is a rotational measure not a linear measure.



Measuring Angles

Degrees

A **degree** is a measurement of angle. One full rotation around the circle is equal to 360 degrees, so one degree is 1/360 of a circle.

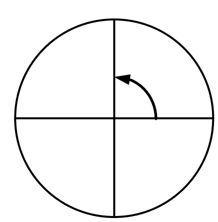
An angle measured in degrees should always include the unit "degrees" after the number, or include the degree symbol °. For example, 90 degrees = $90°$.

Standard Position

When measuring angles on a circle, unless otherwise directed, we measure angles in **standard position**: starting at the positive horizontal axis and with counter-clockwise rotation.

Example 1

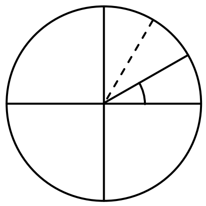
Give the degree measure of the angle shown on the circle.



The vertical and horizontal lines divide the circle into quarters. Since one full rotation is 360 degrees= $360°$, each quarter rotation is 360/4 = $90°$ or 90 degrees.

Example 2

Show an angle of $30°$ on the circle.



An angle of $30°$ is 1/3 of $90°$, so by dividing a quarter rotation into thirds, we can sketch a line at $30°$.

Going Greek

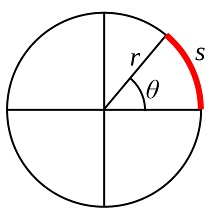
When representing angles using variables, it is traditional to use Greek letters. Here is a list of commonly encountered Greek letters.

θ	φ or ϕ	α	β	γ
theta	phi	alpha	beta	gamma

Angles in Radians

While measuring angles in degrees may be familiar, doing so often complicates matters since the units of measure can get in the way of calculations. For this reason, another measure of angles is commonly used. This measure is based on the distance around a circle.

Arclength

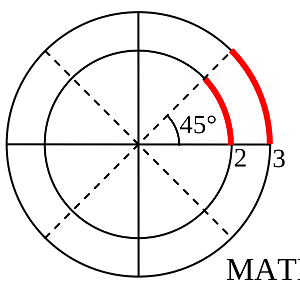


Arclength is the length of an arc, *s*, along a circle of radius *r* subtended (drawn out) by an angle θ. It is the portion of the circumference between the initial and terminal sides of the angle.

The length of the arc around an entire circle is called the circumference of a circle. The circumference of a circle is $C = 2\pi r$.

The ratio of the circumference to the radius, produces the constant 2π. Regardless of the radius, this ratio is always the same, just as how the degree measure of an angle is independent of the radius.

To elaborate on this idea, consider two circles, one with radius 2 and one with radius 3. Recall the circumference (perimeter) of a circle is $C = 2\pi r$, where *r* is the radius of the circle. The smaller circle then has circumference $2\pi(2) = 4\pi$ and the larger has circumference $2\pi(3) = 6\pi$.



Drawing a 45 degree angle on the two circles, we might be interested in the length of the arc of the circle that the angle indicates. In both cases, the 45 degree angle draws out an arc that is 1/8th of the full circumference, so for the smaller circle, the arclength = $\frac{1}{8}(4\pi) = \frac{1}{2}\pi$, and for the larger circle, the length of the arc or arclength = $\frac{1}{8}(6\pi) = \frac{3}{4}\pi$.

Notice what happens if we find the *ratio* of the arclength divided by the radius of the circle:

Smaller circle: $\dfrac{\frac{1}{2}\pi}{2} = \frac{1}{4}\pi$

Larger circle: $\dfrac{\frac{3}{4}\pi}{3} = \frac{1}{4}\pi$

The ratio is the same regardless of the radius of the circle – it only depends on the angle. This property allows us to define a measure of the angle based on arclength.

Radians

The **radian measure** of an angle is the ratio of the length of the circular arc subtended by the angle to the radius of the circle.

In other words, if s is the length of an arc of a circle, and r is the radius of the circle, then

$$\text{radian measure} = \frac{s}{r}$$

If the circle has radius 1, then the radian measure corresponds to the length of the arc.

Because radian measure is the ratio of two lengths, it is a **unitless measure**. It is not necessary to write the label "radians" after a radian measure, and if you see an angle that is not labeled with "degrees" or the degree symbol, you should assume that it is a radian measure.

Considering the most basic case, the unit circle (a circle with radius 1), we know that 1 rotation equals 360 degrees, $360°$. We can also track one rotation around a circle by finding the circumference, $C = 2\pi r$, and for the unit circle $C = 2\pi$. These two different ways to rotate around a circle give us a way to convert from degrees to radians.
1 rotation = $360° = 2\pi$ radians
½ rotation = 180° = π radians
¼ rotation = $90°$ = $\pi/2$ radians

Example 3

Find the radian measure of one third of a full rotation.

For any circle, the arclength along such a rotation would be one third of the circumference, $C = \frac{1}{3}(2\pi r) = \frac{2\pi r}{3}$. The radian measure would be the arclength divided by the radius:

Radian measure = $\frac{2\pi r}{3r} = \frac{2\pi}{3}$.

Converting Between Radians and Degrees

1 degree = $\frac{\pi}{180}$ radians

or: to convert from degrees to radians, multiply by $\frac{\pi \text{ radians}}{180^\circ}$

1 radian = $\frac{180}{\pi}$ degrees

or: to convert from radians to degrees, multiply by $\frac{180^\circ}{\pi \text{ radians}}$

Example 4

Convert $\frac{\pi}{6}$ radians to degrees.

Since we are given a problem in radians and we want degrees, we multiply by $\frac{180^\circ}{\pi}$.

Remember radians are a unitless measure, so we don't need to write "radians."

$\frac{\pi}{6}$ radians = $\frac{\pi}{6} \cdot \frac{180^\circ}{\pi} = 30$ degrees.

Example 5

Convert 15 degrees to radians.

In this example we start with degrees and want radians so we use the other conversion $\frac{\pi}{180^\circ}$ so that the degree units cancel and we are left with the unitless measure of radians.

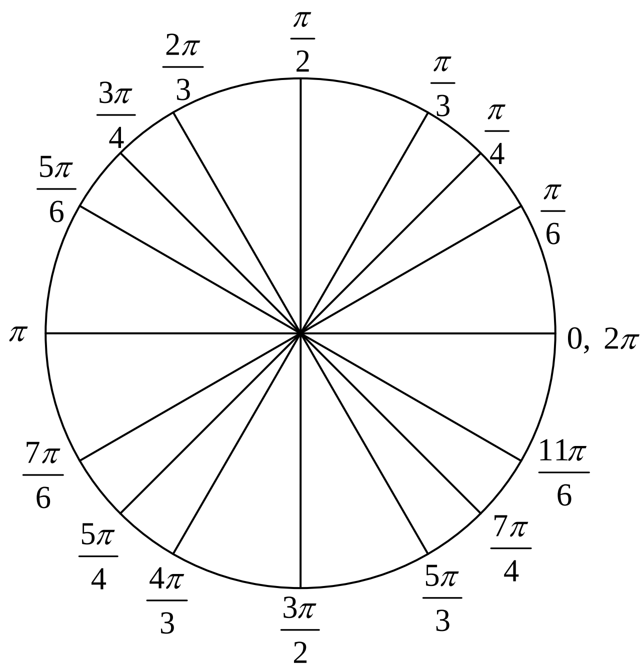
15 degrees = $15^\circ \cdot \frac{\pi}{180^\circ} = \frac{\pi}{12}$

Try it Now

1. Convert $\frac{7\pi}{10}$ radians to degrees.

Just as we listed all the common angles in degrees on a circle, we should also list the corresponding radian values for the common measures of a circle corresponding to degree multiples of 30, 45, 60, and 90 degrees. As with the degree measurements, it would be advisable to commit these to memory.

We can work with the radian measures of an angle the same way we work with degrees.



Arclength and Area of a Sector

Recall that the radian measure of an angle was defined as the ratio of the arclength of a circular arc to the radius of the circle, $\theta = \frac{s}{r}$. From this relationship, we can find arclength along a circle given an angle.

Arclength on a Circle

The length of an arc, s, along a circle of radius r subtended by angle θ in radians is
$s = r\theta$

Example 6

Mercury orbits the sun at a distance of approximately 36 million miles. In one Earth day, it completes 0.0114 rotation around the sun. If the orbit was perfectly circular, what distance through space would Mercury travel in one Earth day?

To begin, we will need to convert the decimal rotation value to a radian measure. Since one rotation = 2π radians,
0.0114 rotation = $2\pi(0.0114) = 0.0716$ radians.
Combining this with the given radius of 36 million miles, we can find the arclength:
$s = (36)(0.0716) = 2.578$ million miles travelled through space.

Try it Now

2. Find the arclength along a circle of radius 10 subtended by an angle of 215 degrees.

In addition to arclength, we can also use angles to find the area of a sector of a circle. A sector is a portion of a circle contained between two lines from the center, like a slice of pizza or pie.

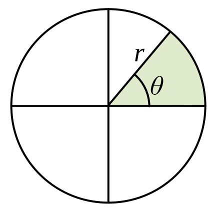
Recall that the area of a circle with radius *r* can be found using the formula $A = \pi r^2$. If a sector is cut out by an angle of θ, measured in radians, then the fraction of full circle that angle has cut out is $\frac{\theta}{2\pi}$, since 2π is one full rotation. Thus, the area of the sector would be this fraction of the whole area:

$$\text{Area of sector } = \left(\frac{\theta}{2\pi}\right)\pi r^2 = \frac{\theta \pi r^2}{2\pi} = \frac{1}{2}\theta r^2$$

Area of a Sector

The **area of a sector** of a circle with radius *r* subtended by an angle θ, measured in radians, is

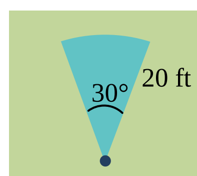
$$\text{Area of sector} = \frac{1}{2}\theta r^2$$



Example 7

An automatic lawn sprinkler sprays a distance of 20 feet while rotating 30 degrees. What is the area of the sector of grass the sprinkler waters?

First we need to convert the angle measure into radians. Since 30 degrees is one of our common angles, you ideally should already know the equivalent radian measure, but if not we can convert:



30 degrees = $30 \cdot \frac{\pi}{180} = \frac{\pi}{6}$ radians.

The area of the sector is then Area $= \frac{1}{2}\left(\frac{\pi}{6}\right)(20)^2 = 104.72 \text{ ft}^2$

Try it Now

3. In central pivot irrigation, a large irrigation pipe on wheels rotates around a center point, as pictured here[1]. A farmer has a central pivot system with a radius of 400 meters. If water restrictions only allow her to water 150 thousand square meters a day, what angle should she set the system to cover?

Try it Now Answers

1. 126°
2. $\frac{215\pi}{18} \approx 37.525$
3. 107.43°

[1] http://commons.wikimedia.org/wiki/File:Pivot_otech_002.JPG

Tutorial – 21

1. Convert the angle 180° to radians.
2. Convert the angle 30° to radians.
3. Convert the angle $\frac{5\pi}{6}$ from radians to degrees.
4. Convert the angle $\frac{11\pi}{6}$ from radians to degrees.
5. On a circle of radius 7 miles, find the length of the arc that subtends a central angle of 5 radians.
6. On a circle of radius 6 feet, find the length of the arc that subtends a central angle of 1 radian.
7. On a circle of radius 12 cm, find the length of the arc that subtends a central angle of 120 degrees.
8. On a circle of radius 9 miles, find the length of the arc that subtends a central angle of 800 degrees.
9. On a circle of radius 6 feet, what angle in degrees would subtend an arc of length 3 feet?
10. On a circle of radius 5 feet, what angle in degrees would subtend an arc of length 2 feet?
11. A sector of a circle has a central angle of 45°. Find the area of the sector if the radius of the circle is 6 cm.
12. A sector of a circle has a central angle of 30°. Find the area of the sector if the radius of the circle is 20 cm.

****

10.2 Right Triangle Trigonometry

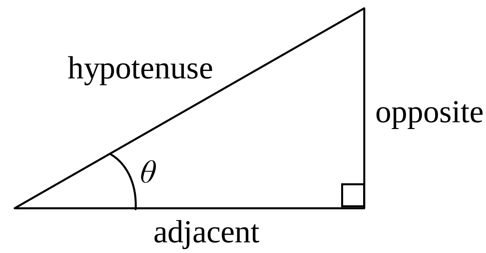
Right Triangle Relationships

Given a right triangle with an angle of θ

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

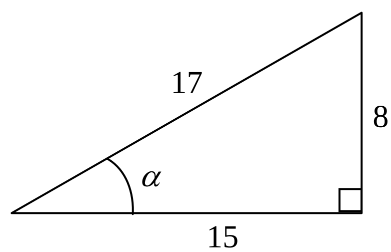
$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$



A common mnemonic for remembering these relationships is SohCahToa, formed from the first letters of "Sine is opposite over hypotenuse, Cosine is adjacent over hypotenuse, Tangent is opposite over adjacent."

Example 1

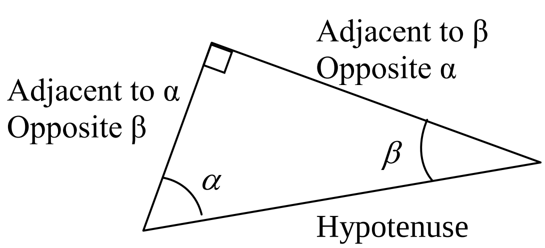
Given the triangle shown, find the value for $\cos(\alpha)$.



The side adjacent to the angle is 15, and the hypotenuse of the triangle is 17, so

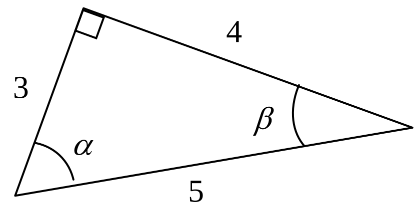
$$\cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{15}{17}$$

When working with general right triangles, the same rules apply regardless of the orientation of the triangle.
In fact, we can evaluate the sine and cosine of either of the two acute angles in the triangle.



Example 2

Using the triangle shown, evaluate $\cos(\alpha)$, $\sin(\alpha)$, $\cos(\beta)$, and $\sin(\beta)$.



$$\cos(\alpha) = \frac{\text{adjacent to } \alpha}{\text{hypotenuse}} = \frac{3}{5}$$

$$\sin(\alpha) = \frac{\text{opposite} \alpha}{\text{hypotenuse}} = \frac{4}{5}$$

$$\cos(\beta) = \frac{\text{adjacent to } \beta}{\text{hypotenuse}} = \frac{4}{5}$$

$$\sin(\beta) = \frac{\text{opposite} \beta}{\text{hypotenuse}} = \frac{3}{5}$$

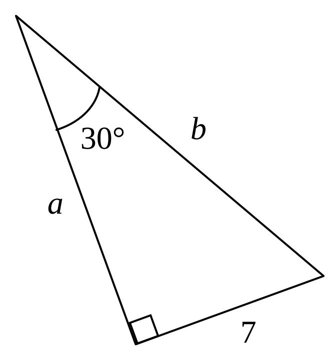
Try it Now

1. A right triangle is drawn with angle α opposite a side with length 33, angle β opposite a side with length 56, and hypotenuse 65. Find the sine and cosine of α and β.

In the previous examples we evaluated the sine and cosine on triangles where we knew all three sides of the triangle. Right triangle trigonometry becomes powerful when we start looking at triangles in which we know an angle but don't know all the sides.

Example 3

Find the unknown sides of the triangle pictured here.



Since $\sin(\theta) = \dfrac{\text{opposite}}{\text{hypotenuse}}$,

$$\sin(30°) = \frac{7}{b}$$

From this, we can solve for the side *b*.

$$b\sin(30°) = 7$$

$$b = \frac{7}{\sin(30°)}$$

To obtain a value, we can evaluate the sine and simplify

$$b = \frac{7}{1/2} = 14$$

To find the value for side *a*, we could use the cosine, or simply apply the Pythagorean Theorem:

$a^2 + 7^2 = b^2$

$a^2 + 7^2 = 14^2$

$a = \sqrt{147}$

Notice that if we know at least one of the non-right angles of a right triangle and one side, we can find the rest of the sides and angles.

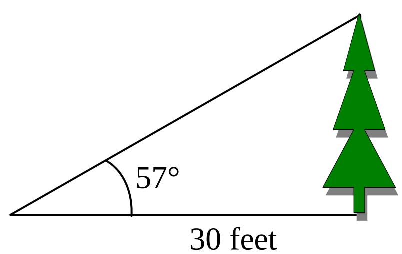
Try it Now

2. A right triangle has one angle of $\frac{\pi}{3}$ and a hypotenuse of 20. Find the unknown sides and angles of the triangle.

Example 4

To find the height of a tree, a person walks to a point 30 feet from the base of the tree, and measures the angle from the ground to the top of the tree to be 57 degrees. Find the height of the tree.

We can introduce a variable, *h*, to represent the height of the tree. The two sides of the triangle that are most important to us are the side opposite the angle, the height of the tree we are looking for, and the adjacent side, the side we are told is 30 feet long.



The trigonometric function which relates the side opposite of the angle and the side adjacent to the angle is the tangent.

$\tan(57°) = \frac{\text{opposite}}{\text{adjacent}} = \frac{h}{30}$ Solving for *h*,

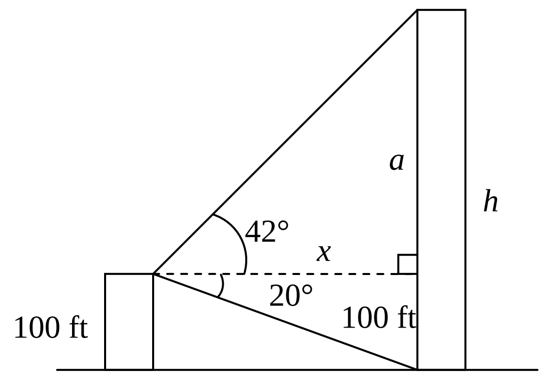
$h = 30\tan(57°)$ Using technology we can approximate a value

$h = 30\tan(57°) \approx 46.2$ feet

The tree is approximately 46 feet tall.

Example 5

A person standing on the roof of a 100 foot building is looking towards a skyscraper a few blocks away, wondering how tall it is. She measures the angle of declination from the roof of the building to the base of the skyscraper to be 20 degrees and the angle of inclination to the top of the skyscraper to be 42 degrees.



To approach this problem, it would be good to start with a picture. Although we are interested in the height, *h*, of the skyscraper, it can be helpful to also label other unknown quantities in the picture – in this case the horizontal distance *x* between the buildings and *a*, the height of the skyscraper above the person.

To start solving this problem, notice we have two right triangles. In the top triangle, we know one angle is 42 degrees, but we don't know any of the sides of the triangle, so we don't yet know enough to work with this triangle.

In the lower right triangle, we know one angle is 20 degrees, and we know the vertical height measurement of 100 ft. Since we know these two pieces of information, we can solve for the unknown distance *x*.

$$\tan(20°) = \frac{\text{opposite}}{\text{adjacent}} = \frac{100}{x} \qquad \text{Solving for } x$$

$$x\tan(20°) = 100$$

$$x = \frac{100}{\tan(20°)}$$

Now that we have found the distance *x*, we know enough information to solve the top right triangle.

$$\tan(42°) = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{x} = \frac{a}{100/\tan(20°)}$$

$$\tan(42°) = \frac{a\tan(20°)}{100}$$

$$100\tan(42°) = a\tan(20°)$$

$$\frac{100\tan(42°)}{\tan(20°)} = a$$

Approximating a value,

$$a = \frac{100\tan(42°)}{\tan(20°)} \approx 247.4 \text{ feet}$$

Adding the height of the first building, we determine that the skyscraper is about 347 feet tall.

Try it Now Answers

1. $\sin(\alpha) = \frac{33}{65}$ $\cos(\alpha) = \frac{56}{65}$ $\sin(\beta) = \frac{56}{65}$ $\cos(\beta) = \frac{33}{65}$

2. $\cos\left(\frac{\pi}{3}\right) = \frac{\text{adjacent}}{\text{hypoteuse}} = \frac{\text{Adj}}{20}$ so, adjacent $= 20\cos\left(\frac{\pi}{3}\right) = 20\left(\frac{1}{2}\right) = 10$

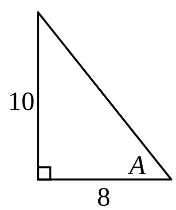
$\sin\left(\frac{\pi}{3}\right) = \frac{\text{Opposite}}{\text{hypoteuse}} = \frac{\text{Opp}}{20}$ so, opposite$= 20\sin\left(\frac{\pi}{3}\right) = 20\left(\frac{\sqrt{3}}{2}\right) = 10\sqrt{3}$

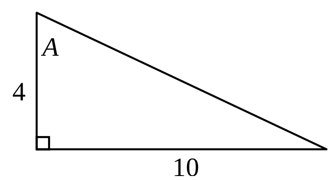
Missing angle = 30 degrees or $\pi/6$

Tutorial – 22

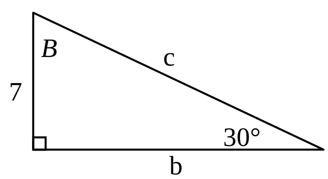
Note: pictures may not be drawn to scale.

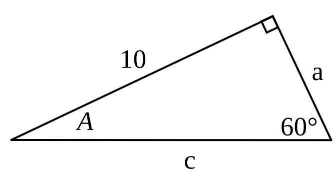
In each of the triangles below, find $\sin(A), \cos(A), \tan(A), \sec(A), \csc(A), \cot(A)$.

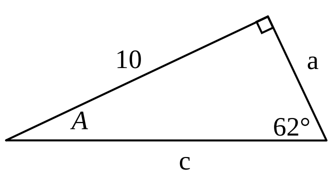
1. 

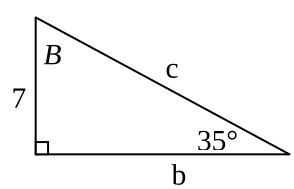
2. 

In each of the following triangles, solve for the unknown sides and angles.

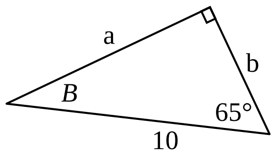
3. 

4. 

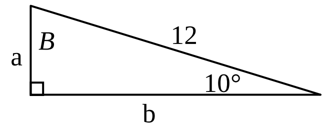
5. 

6. 

7.



8.



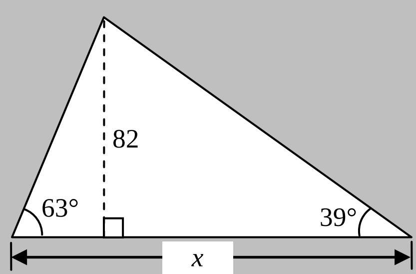
9. A 33-ft ladder leans against a building so that the angle between the ground and the ladder is 80°. How high does the ladder reach up the side of the building?

10. A 23-ft ladder leans against a building so that the angle between the ground and the ladder is 80°. How high does the ladder reach up the side of the building?

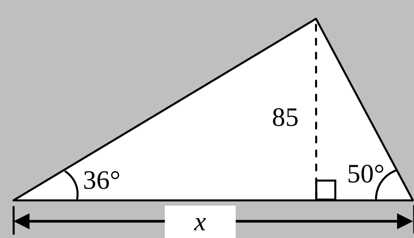
11. The angle of elevation to the top of a building in New York is found to be 9 degrees from the ground at a distance of 1 mile from the base of the building. Using this information, find the height of the building.

12. The angle of elevation to the top of a building in Seattle is found to be 2 degrees from the ground at a distance of 2 miles from the base of the building. Using this information, find the height of the building.

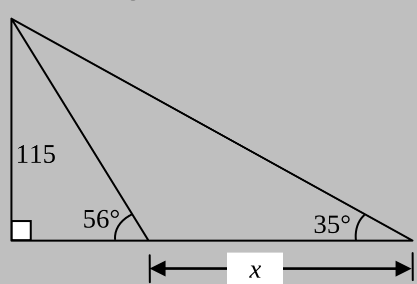
13. Find the length *x*.



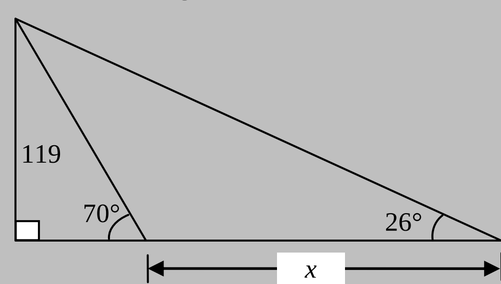
14. Find the length *x*.



15. Find the length *x*.



16. Find the length *x*.



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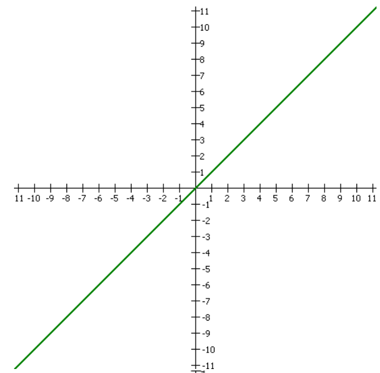
APPENDIX 1- Graphs of some important functions/equations

1 **Identity Function**

$f(x) = x$

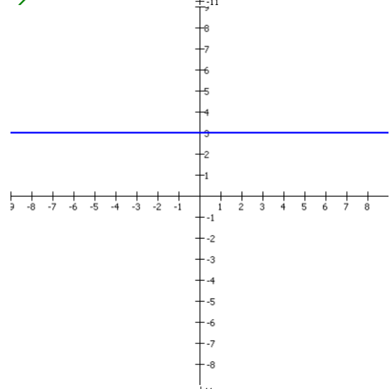
$y = x$



2 **Constant Function**

$f(x) = \text{c}$

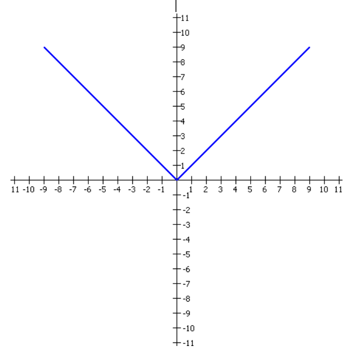
Eg: y = 3



3 **Modulus Function**

$f(x) = | x |$

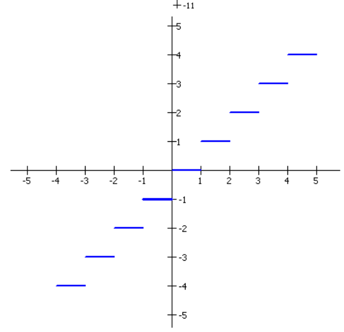
$y = | x |$



4 **Greatest Integer Function**

$f(x) = [x]$

$y = [x]$



5 **Linear Functions**

f(x) = a*x* + b, a > 0

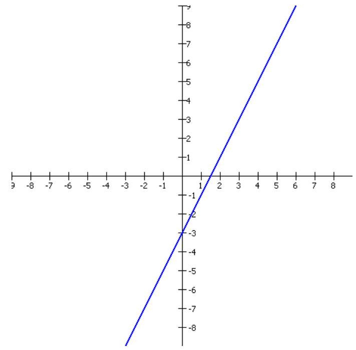
Eg: *f(x)* = 2*x* – 3

Or

y = 2*x* – 3

Or

2*x* – *y* + 3 = 0



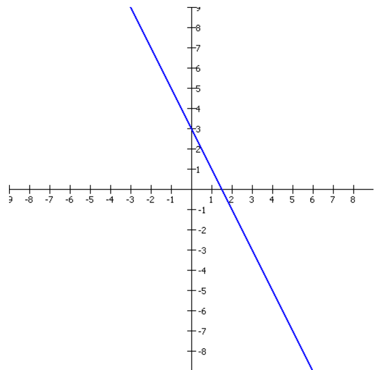
6 **Linear Functions**

f(x) = a*x* + b, a < 0

Eg: *f(x)* = 3 – 2*x*

Or

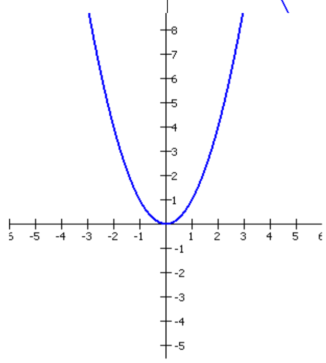
2*x* + *y* = 3



7 **Quadratic Functions**

$f(x) = x^2$

$y = x^2$

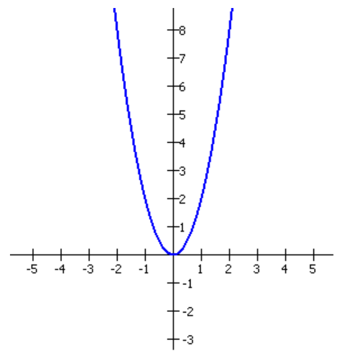


8 **Quadratic Functions**

$f(x) = ax^2$, a>0

Eg 1: $f(x) = 2x^2$

$y = 2x^2$

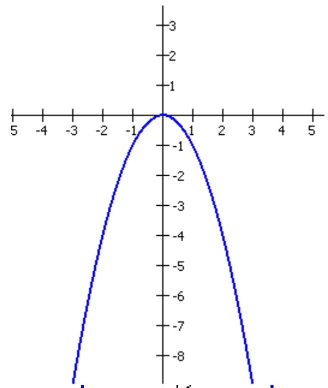


9 **Quadratic Functions**

$f(x) = ax^2$, $a<0$

Eg: $f(x) = - x^2$

$y = - x^2$

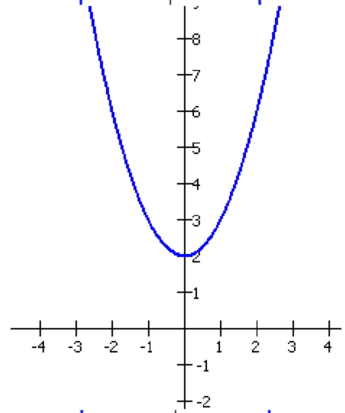


10 **Quadratic Functions**

$f(x) = ax^2 + b$, $a>0$

Eg 1: $f(x) = x^2 + 2$

$y = x^2 + 2$

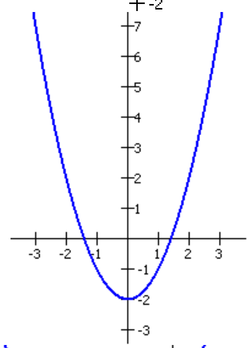


11 **Quadratic Functions**

$f(x) = ax^2 + b$, $a>0$

Eg 2: $f(x) = x^2 – 2$

$y = x^2 – 2$

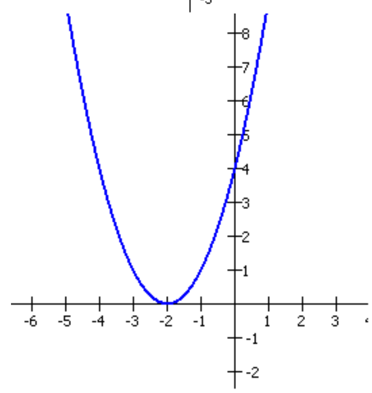


12 **Quadratic Functions**

$f(x) = (ax + b)^2$, $a>0$

Eg 1: $f(x) = (x + 2)^2$

$y = (x + 2)^2$

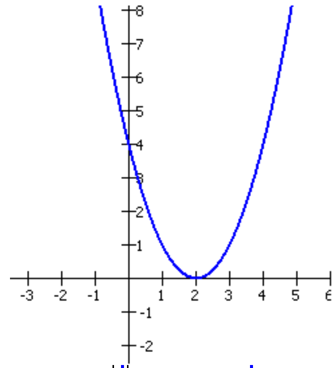


13 **Quadratic Functions**

$f(x) = (ax + b)^2$, $a>0$

Eg 2: $f(x) = (x - 2)^2$

$y = (x - 2)^2$



14 **Quadratic Functions**

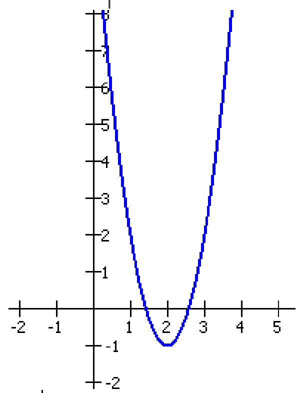
$f(x) = ax^2+bx+c$, $a>0$

Eg 1:

$f(x) = 3(x - 2)^2 – 1$

Or

$y = 3x^2 – 12x + 11$



15 **Quadratic Equations in *y***

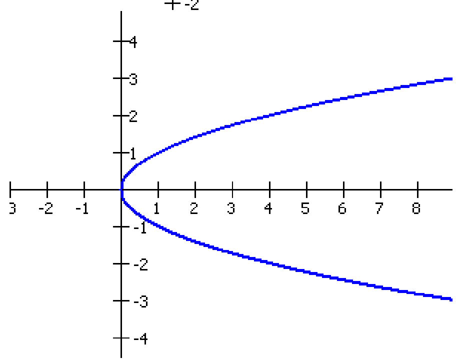
$y^2 = 4ax$, $a>0$

Eg:$f(x) = \sqrt{x}$

$y = \sqrt{x}$

Or

$y^2 = x$



16 **Quadratic Equations in *y***

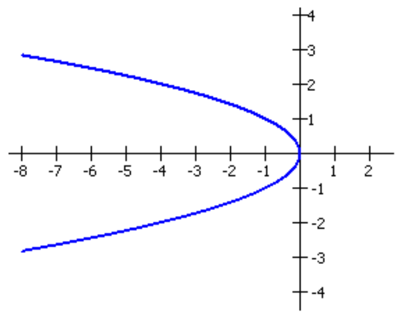
$y^2 = 4ax$, $a<0$

Eg:$f(x) = -\sqrt{x}$

$y = -\sqrt{x}$

Or

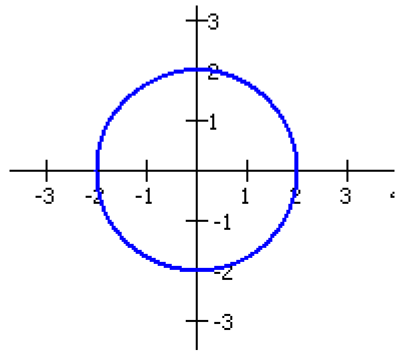
$y^2 = - x$



17 **Circle**

$x^2 + y^2 = a^2$

Eg: $x^2 + y^2 = 4$



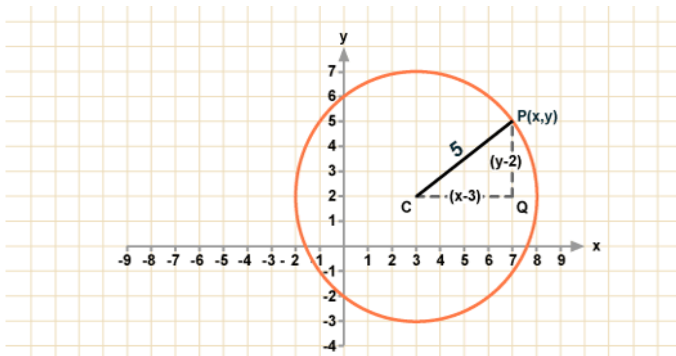
18 **Circle**

$x^2 + y^2 + 2hx + 2ky + c = 0$

Eg: $(x - 3)^2 + (y - 2)^2 = 25$

Or

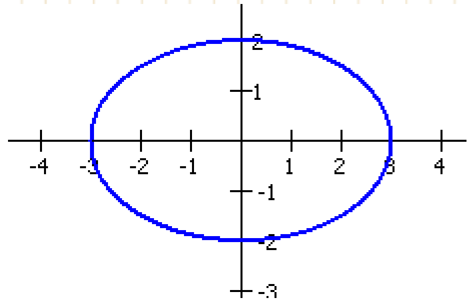
$x^2 + y^2 - 6x - 4y - 25 = 0$



19 **Ellipse**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

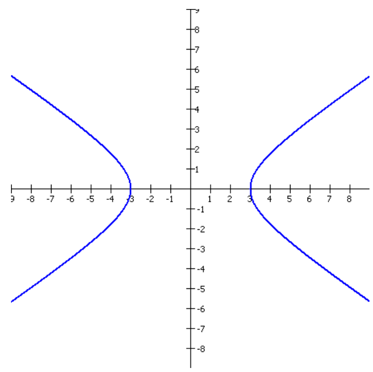
Eg: $\frac{x^2}{9} + \frac{y^2}{4} = 1$



20 **Hyperbola**

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

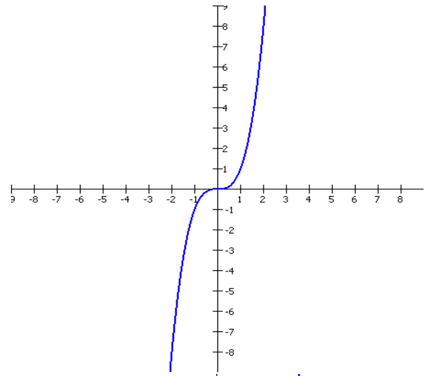
Eg: $\frac{x^2}{9} - \frac{y^2}{4} = 1$



21 **Cubic polynomial**

Eg :$f(x) = x^3$

$y = x^3$

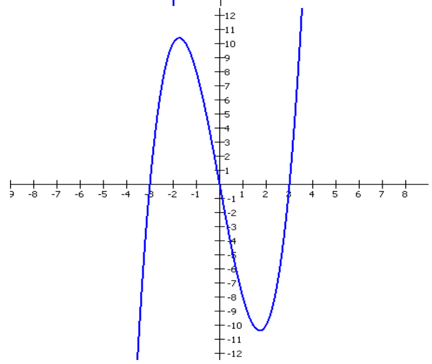


22 **Cubic polynomial**

$f(x) = ax^3 + bx^2 + cx + d, a > 0$

Eg: $f(x) = x^3 – 9x$

$y = x(x + 3)(x - 3)$

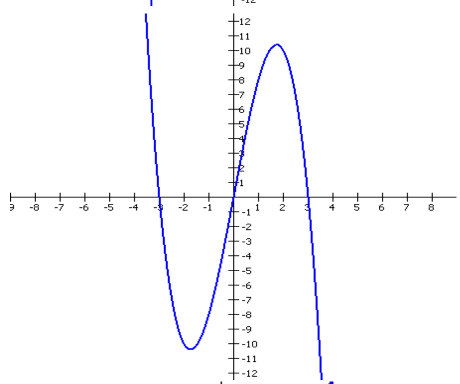


23 **Cubic polynomial**

$f(x) = ax^3 + bx^2 + cx + d, a < 0$

Eg: $f(x) = - x^3 + 9x$

$y = - x(x + 3)(x - 3)$

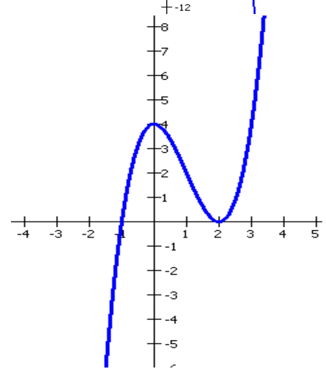


24 **Cubic polynomial**

$f(x) = ax^3 + bx^2 + cx + d, a> 0$

Eg:$f(x) = x^3- 3x^2 + 4$

$y = (x + 1)(x - 2)^2$



APPENDIX 2 – List of Mathematical Symbols

Basic Arithmetic operations

Symbol	Read as	Name	Result
$+$	Plus	Addition	Sum
$-$	Minus	Subtraction	Difference
$\times$	Multiply / times	Multiplication	Product
$\div$	Divided by	Division	Quotient

Other Symbols

Symbol	Read as	Meaning / Example
$=$	Equal to	Eg: $3 + 5 = 8$
$\neq$	Not equal to	Eg: $10 - 6 \neq 5$
$\approx$	Approximately equal to	Eg: $2 \div 3 \approx 0.667$
$<$	Less than	Eg: $5 \times 7 < 57$
$\leq$	Less than or equal to	Eg: $24 \div 6 \leq 4$
$>$	Greater than	Eg: $54 > 90 - 40$
$\geq$	Greater than or equal to	Eg: $4 \times 4 \geq 8$
$\nless$	Not less than	Eg: $10 + 5 \nless 12$
%	Percentage	
∞	Infinity	The largest value
$-\infty$	Minus infinity	The smallest value.
$\bar{x}$	x bar	Eg: $3.\bar{5} = 3.5555\ldots.$
$\lvert x \rvert$	Modulus / Absolute value of x	Eg: $\lvert -2 \rvert = 2$, $\lvert 3 \rvert = 3$
( )	Parentheses / Simple bracket / Round bracket / Open bracket	
[ ]	Square bracket / Box bracket / Closed bracket	
{ }	Braces / Curly bracket	
a^b	Exponential form	$a^b = a \times a \times a \times \ldots \times a$ (b times)

Here, *a* is the base and *b* is the power.
We read a^b as *a raised to b.*

	Eg: $2^3 = 2 \times 2 \times 2 = 8$	
a^2	a square	$a^2 = a \times a$ Eg: $3^2 = 3 \times 3 = 9$
a^3	a cubed	$a^3 = a \times a \times a$ Eg: $5^3 = 5 \times 5 \times 5 = 125$
$\sqrt{\ }$	Root	Eg: $\sqrt{25} = 5$, $\sqrt{a^2} = a$
$\frac{a}{b}$	Fraction	$\frac{a}{b} = a \div b$
	Here, *a* is the numerator and *b* is the denominator. We read $\frac{a}{b}$ as *a over b* or *a by b.*	
π	Pie	$\pi \approx 3.14$ or $\pi \approx \frac{22}{7}$

BODMAS Rule

We use this rule to simplify calculations involving different arithmetic operations.

Bracket
Of
Division
Multiplication
Addition
Subtraction

Eg: Simplify: $(2 \times 15 \div 3 - 4) + 24 \div (10 - 4)$

Ans: $(2 \times 15 \div 3 - 4) + 24 \div (10 - 4)$

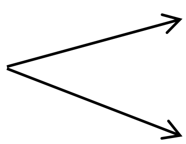
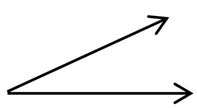
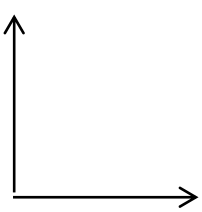
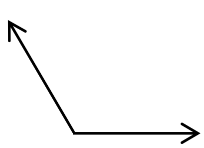
$= (2 \times 5 - 4) + 24 \div 6 \quad (10 – 4) + 4$

$= 6 + 4 \qquad = \mathbf{10}$

Set theoretical Notations

Symbol	Read as	Meaning / Example
{ }	Set	A = {1, 3, 5, 7}, B = { 4, 5, 6}
$\in$	belongs to	$1 \in A$
$\notin$	does not belong to	$0 \notin A$
| A |	Cardinality of A	| A | = 4
$\cup$	Union	$A \cup B = \{1, 3, 4, 5, 6, 7\}$
$\cap$	Intersection	$A \cap B = \{5\}$
–	Difference (*minus*)	A- B = {1, 3, 7}
$\varnothing$	Null set (*phi*)	Set of all even numbers in A is $\varnothing$

Basic geometrical Shapes on a plane

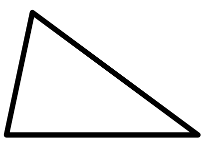
Symbol	Name	Definition
•	Point	*A point is the smallest unit in a plane. It can be defined as a circle with radius zero. Normally denoted by the letters of English alphabet such as P, Q, A, B etc.*
	Line	*A line has no end points and therefore it has infinite length.*
	Ray	*A can be extended infinitely to one direction. It has one terminal point. Also, the length of a ray is infinite.*
	Line segment	*It is the part of a line and it has fixed length.*
	**Angles**	*An angle is a shape formed when the terminal points of two rays coincide.*
	Acute Angle	*An acute angle is an angle that measures less than* 90^0.
	Right angle	*A right angle is an angle which measures exactly* 90^0.
	Obtuse angle	*An angle that measures more than* 90^0 *but less than* 180^0 *is known as an obtuse angle.*

$||$ Parallel — *Two lines are said to be parallel if the distance between them is same everywhere. They do not intersect.*

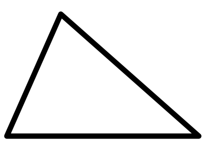
$\perp$ Perpendicular — *Two lines are said to be perpendicular if the angle between them is* 90^0.

<u>Polygons</u> — *Polygons are closed plane figures with three or more edges.*

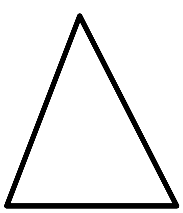
Triangles — *A triangle is a polygon with three sides. The sum of all the angles of a triangle is two right angles. i.e,* 180^0.
The sum of any two sides of a triangle will be greater than the third side.



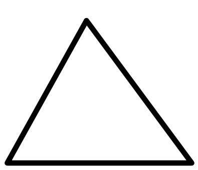
Scalene triangle — *A scalene triangle is a triangle with all the three sides of different lengths. All the three angles of a scalene triangle will also be different.*

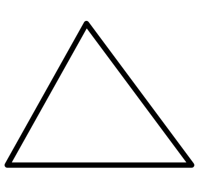


Isosceles triangle — *An isosceles triangle is a triangle having two sides of equal length. The angles opposite to these sides are also equal.*

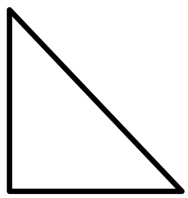


Equilateral triangle — *A triangle with all three sides equal is called an equilateral triangle. Each angle of an equilateral triangle is* 60^0.

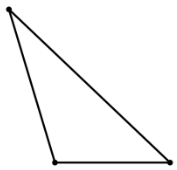




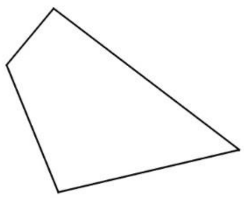
Acute triangle — *A triangle having all the three angles measure less than* 90^0 *(acute) is called an acute triangle.*



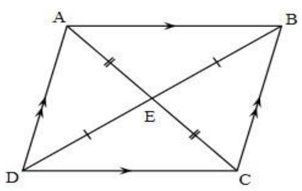
Right triangle — *A right triangle is a triangle with one of its angles measures 90^0(right angle).*



Obtuse triangle — *A triangle whose one angle is more than 90^0(acute angle) is called an obtuse trangle.*



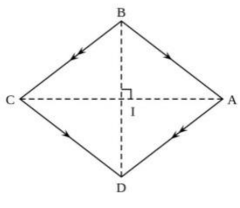
Quadrilaterals — *A quadrilateral is a polygon with four sides and four vertices. The sum of all the angles of a triangle is two straight angles. i.e, 360^0. The line segments joining the opposite vertices of a quadrilateral are called its diagonals.*



Parallelogram — *A parallelogram is a quadrilateral whose opposite sides are parallel. Opposite angles of a parallelogram are equal. Its diagonals bisect each other. i.e, in the given figure, AB || DC, AD || BC,*

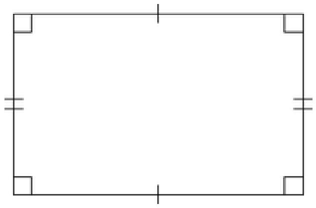
$\angle ABC = \angle ADC$ and $\angle DAB = \angle DCB$

Also, AB = CD, AD =BC, AE = CE and BE = DE.

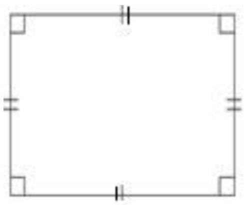


Rhombus — *A rhombus is a parallelogram with all the four sides equal. The diagonal of a rhombus bisect each other at 90^0.*

In the figure, AB = BC = CD = DA and $AC \perp BD$.

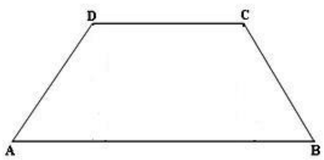


Rectangle — *A rectangle is a parallelogram with one of the angles (or all the angles) measures 90^0. The diagonals of a rectangle are equal in length.*



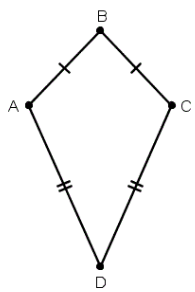
Square

A square is a rhombus with one of the angles (or all the angles) measures 90^0. The diagonals of a rectangle are equal in length.



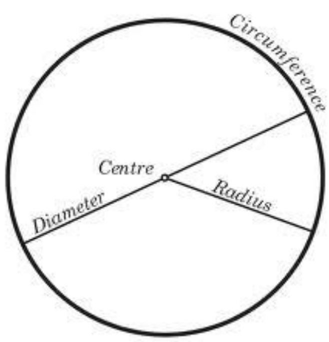
Trapezium

A quadrilateral with exactly one pair of opposite sides are parallel are called a trapezium.
In the given figure, AB||CD. But, AD is not parallel to BC.



Kite

A kite is a quadrilateral whose two pairs of adjacent sides are equal. In the given figure, AB = BC and AD = DC.

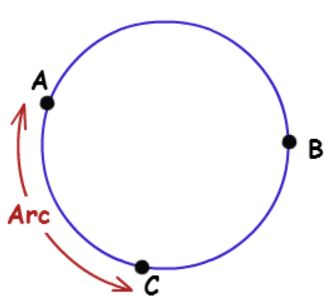


Circle

A circle is a collection of all points located at a fixed distance from a fixed point. The fixed distance is called radius and the fixed point is called its centre. A line segment joining any two pints of a circle is called a chord. A chord passing through the centre of a circle is called a diameter.

For a circle, diameter = 2 × radius.

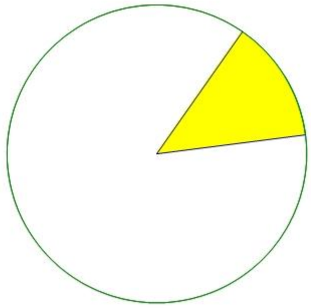
The total length of the boundary of a circle is known as its circumference.



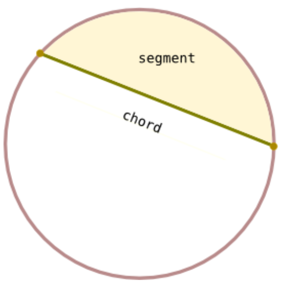
Arc

An arc of a circle is a part of its circumference. A semicircle is an arc whose length is half of the circumference. An arc whose length is less than a semicircle is known as a minor arc whereas the arc whose length is more than that of a semicircle is called a major arc. In the given figure,

AB is a minor arc   ABC is a major arc.



Sector *A region bounded by two radii and an arc as given in the figure is called a sector.*



Segment *A region bounded by a chord and an arc as given in the figure is called a sector.*

Answers for selected questions

Tutorial – 1

1)	$\frac{5}{4}$	2)	$\frac{13}{30}$	3)	$\frac{2}{35}$	4)	$\frac{-44}{45}$	5)	$\frac{-1}{8}$
6)	$\frac{4}{3}$	7)	-40	8)	$\frac{1}{4}$	9)	$\frac{69}{100}$	10)	$\frac{5}{36}$
11)	$\frac{19}{24}$	12)	$\frac{75}{8}$	13)	$\frac{-1}{11}$	14)	$\frac{-29}{18}$	15)	-6
16)	$\frac{17}{3}$	17)	$\frac{4}{13}$						

II Rational : *a, b, d, e, f, h, i, k, l, n , o*

Irrational : *c, g, j , m*

Tutorial – 2

1)	$<$	2)	$>$	3)	$=$	4)	$=$		
5)	$\{1,2,3,4,5,6,7\}$	6)	$\{4,5,6\}$	7)	$\{6\}$	8)	$\{6, 7\}$		
9)	$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$			10)	$\{1, 2, 3, 4, 5, 6, 7\}$				
11)	$\{4, 5, 6, 7, 8, 9\}$			12)	$\{6\}$			13)	$x < 0$
14)	$-3 \le x \le 0$	15)	$-7 < x \le 8$	16)	$-10 \le x \le 10$			17)	$4 \le x < 9$
18)	[2, 7)	19)	(-1, 6]	20)	[-2, 2]			21)	$(-\infty, -1.5)$
22)	$(3, \infty)$	23)	$(-\infty, 0)$	24)	4	25)	-7	26)	6
27)	$12 - \pi$	28)	0	29)	100	30)	17	31)	28
32)	$\frac{73}{10}$	33)	0.8						

Tutorial – 3

1)	-8	2)	-8	3)	1	4)	36	5)	81
6)	$\frac{1}{16}$	7)	27	8)	$\frac{1}{36}$	9)	a^5	10)	x^9
11)	y^2	12)	b^3	13)	$\frac{1}{p^2}$	14)	$\frac{1}{q}$	15)	x^{10}
16)	a^{12}	17)	$a^6 b^3$	18)	$x^{12} y^8$	19)	$\frac{a^{12}}{b^{15}}$	20)	$\frac{x^{15}}{y^{20}}$

21)	$\frac{b^3}{a^2}$	22)	p^4q^3	23)	$a^{3x}b^{4y}$	24)	$a^{4x-2y+z}$	25)	$-12a^6b^6c$
26)	$a^2(a+b)^3$	27)	$\frac{x^6}{y^{17}}$	28)	$-27a^{11}b^{13}c^5$	29)	$\frac{c^2d^2e^4}{b^2}$		
30)	$\frac{2c^7}{9b}$	31)	4	32)	$\left(\frac{9}{19}\right)^2$	33)	$\frac{9}{4}$	34)	$(-2)^4$

Tutorial – 4

1)	$5^{\frac{1}{3}}$	2)	$3^{-\frac{1}{2}}$	3)	$5^{\frac{3}{4}}$	4)	$\sqrt{2^5}$	5)	$\frac{1}{\sqrt[3]{3^2}}$
6)	$\sqrt[3]{2}$	7)	2	8)	-3	9)	$\frac{1}{5}$	10)	$\frac{2}{3}$
11)	$-\frac{1}{4}$	12)	4	13)	$6\sqrt{3}+10\sqrt{2}$	14)	$3\sqrt{5}+7\sqrt{2}$		
15)	$7\sqrt{2}$	16)	$\sqrt{5}$	17)	$\frac{5\sqrt{3}}{3}$	18)	$5\sqrt{2}$		
19)	$2\sqrt{13}$	20)	$5\sqrt{5}$	21)	$3\sqrt{15}$				

Tutorial – 5

I. 1) $3,\ 5a, -3b, 8$ 2) $1,\ 12a^2b$ 3) $2,\ -5x^2y, 2y^2$

4) $5,\ 2x, -x^2, 3x^3, -4x^5 - 7$ 5) 1, 7 6) $2,\ x^3, x$

7) $4,\ 2y, 2xy, 2x, -1$ 8) $1,\ -m^3n$

II. 1) 4, -2, 3, -7 2) $-3, \frac{1}{2}, 0, 5$ 3) $0, 0, 1, -9$

4) $0, 1, -1, 0$ 5) $-1, 0, 0, 0$ 6) $1.5, -2.5, 0.5, 0$

III. 1) 4 2) -3 3) 0

IV. 1) -2 2) 0 3) -8

V. 1) $3a+3b$ 2) $-10a$ 3) $26a+b$ 4) $2x^2+x-8$

5) x^2-2x+4 6) $3xy-2y^2+xy^2-x^2y$ 7) $2a-7b$

8) $3x+1$ 9) $6a^2-9$ 10) $5a-10b$

11) $-2x^3-x^2+3x-2$ 12) $-2x-2y$ 13) $x^2-y^2+2xy-x^2y-xy^2$

14) 0 | 15) $15t^2 - 22t + 8$ | 16) $12x^2 + 17x - 5$

17) $3x^2 - 5xy - 2y^2$ | 18) $8x^2 + 14xy - 15y^2$ | 19) $x^5 + 4x^4 - 11x^2 - 2x + 6$

20) $x^5y + x^3y^2 - x^2y^3 - x^3y^3 - xy^4 + y^5$

Tutorial – 6

1) $4 + 4x + x^2$
2) $4a^2 + 12a + 9$
3) $9 - 6y + y^2$
4) $25 - 20x^2 + 4x^4$
5) $1 + 4y + 4y^2$
6) $9x^2 - 24x + 16$
7) $4x^4 + 12x^2y^2 + 9y^4$
8) $c^2 - 2 + \frac{1}{c^2}$
9) $a^2x^2 + 2abxy + b^2y^2$
10) $4a^2b^4 - 12ab^2c^3d^4 + 9c^6d^8$
11) $x^2 + 8xy + 4y^2$
12) $x^2 + 4y^2$
13) $2a^2 + 8$
14) $8a$
15) $x^2 - 4y^2$
16) $4x^2 - 9y^2$
17) $x^4 - a^4$
18) $x^4 - y^4$
19) 6
20) 19
21) 52
22) 29
23) 11
24) 2
25) 17
26) 13

Tutorial – 7

1) $3x + 5$
2) $15x^2$
3) $2x^2 - 3$
4) $- 12x^4$
5) $\frac{4}{x} + \frac{2x}{y}$
6) $p - q + pr$
7) $- x + y + z$
8) $- x^3yz$
9) $x^2(x - 1)$
10) $3x(x - 2)$
11) $-2x(x^2 - 8)$
12) $x(x^2 - x + 1)$
13) $2xy^2(3xy - 5)$
14) $2x^2(x^2 + 2x - 7)$
15) $2xy(2x^2 - 3y^2 + 4xy)$
16) $x^2y^3(x^2 + x - 5)$
17) $(x + a)(x + b)$
18) $(x^2 + 1)(a + b)$
19) $(x - 1)(x^2 + 1)$
20) $(x - 2)(x^2 - 3)$
21) $(a - b)(a - c)$
22) $(a - b)(a + c)$
23) $(x + 3)(x + 1)$
24) $(x + 4)(x + 3)$
25) $(x - 3)(x - 2)$
26) $(x - 6)(x - 2)$
27) $(x - 5)(x - 3)$
28) $(p + 12)(p - 3)$
29) $(x - 5)(x + 3)$
30) $(r - 22)(r + 1)$
31) $x(x - 8)(x + 5)$
32) $x^2(x - 2)(x + 1)$
33) $(x + 10)^2$
34) $(x + 6)^2$
35) $(3x + 4y)^2$
36) $(2x - 3)^2$
37) $(m - 2n)^2$
38) $(7a + 4b)(7a - 4b)$
39) $(4x + 5y)(4x - 5y)$
40) $(9 + 7y)(9 - 7y)$
41) $(x^2 + 9y^2)(x + 3y)(x - 3y)$
42) $(x^2 + 3y)(x^2 - 3y)$

Tutorial – 8

1)	$\frac{x-1}{x+1}$	2)	$\frac{x+2}{2x-2}$	3)	$\frac{x-3}{x+7}$	4)	$\frac{x+2}{x+1}$	5)	$\frac{x+1}{3x+6}$
6)	$\frac{a}{a-1}$	7)	$\frac{x^2+4x-4}{x^2-4}$	8)	$\frac{3x+2}{x+1}$	9)	$\frac{5x}{x^2-x-6}$		
10)	$\frac{2x^2}{x^2-1}$	11)	$\frac{2}{x^2-1}$	12)	$\frac{x-4}{x^2+x-2}$	13)	$\frac{x^2-x+12}{x^2-2x-8}$		
14)	$\frac{1}{5x-10}$	15)	$\frac{x+3}{x-3}$	16)	$\frac{1}{a^2+6a+9}$	17)	$\frac{x+4}{x-3}$		
18)	x^3+x^2	19)	x^2+2x+1	20)	$\frac{x^2-4x+4}{x+1}$				

Tutorial – 9

1)	4	2)	12	3)	$-11y$	4)	$4a + 5b$	5)	-12
6)	10	7)	60	8)	a^5	9)	$\frac{b^2}{a^2}$	10)	ab^4
11)	0	12)	18	13)	$2b + 3$	14a)	$x = 2$ is a solution		
14b)	$x = -2$ is not a solution			15a)	$x = 2$ is a solution				
15b)	$x = 4$ is not a solution			16)	- 16	17)	0	18)	- 12
19)	$\frac{7}{2}$	20)	8	21)	- 3	22)	- 21	23)	$2b + 2$
24)	5	25)	$a^2b + b^2 + a^2 + b$			26)	$R = \frac{PV}{nT}$		
27)	$G = \frac{r^2F}{mM}$	28*i*)	$a = \frac{v-u}{t}$	28*ii*)	$a = \frac{2s}{t^2} - \frac{2u}{t}$	28*iii*)	$a = \frac{v^2-u^2}{2s}$		
29)	$\frac{13}{20}$	30)	$\frac{5}{14}$	31)	- 3	32)	$\frac{13}{6}$	33)	$\frac{1}{3}$
34)	$-\frac{6}{7}$	35)	6						

Tutorial – 10

1)	$x = \pm 8$	2)	$x = \pm 3$	3)	$x = \pm\frac{6}{5}$	4)	$x = \pm\sqrt{\frac{52}{7}}$
5)	$x = 1 \, and \, -4$	6)	$x = -3 \, and \, -5$			7)	$x = -10 \, and \, -20$

8) $x = \frac{3}{2}$ *and* 2 9) $x = -\frac{2}{5}$ *and* 2 10) $x = 3 \pm \sqrt{8}$

11) $x = \frac{-1 \pm \sqrt{6}}{2}$ 12) *No real roots* 13) $y = \frac{2 \pm \sqrt{14}}{5}$

14) $x = 3 \, and -4$ 15) $x = \frac{5 \pm \sqrt{73}}{2}$ 16) $x = -1 \pm \sqrt{28}$

17) *D = 36; two real roots* 18) *D = 0; one real root*

19) *D = -23; no real roots* 20) *D = 225; two real roots*

21) *D = 49; two real roots*

Tutorial – 11

1) *8, 9, 10* 2) *21, 23,25,27* 3) *46* 4) *48*

5) *20, 21, 22* 6) *12, 14, 16, 18* 7) *45m*

8) *-33 and - 34* 9) *11 and 121& -12 and 144* 10) *14 and 16 & - 14 and - 16*

11) *Width = 22m and length =25m* 12) *Width = 16 inches and length = 35 inches*

Tutorial – 12

1*i*) C 1*ii*) B 1*iii*) C

2) (*4*, ∞) 3) (- ∞, - *2*) 4) (-∞, *2*]

5) $[7, \infty)$ 6) $\left(-\infty, \frac{5}{2}\right)$ 7) $\left[\frac{2}{3}, \infty\right)$

8) $[1, \infty)$ 9) $(-\infty, -1]$ 10) $\left(\frac{16}{3}, \infty\right)$

11) $\left(-\infty, -\frac{1}{3}\right)$ 12) [-3, -1] 13) [3, 6]

14) [-2, 2) 15) (-1, 3]

Tutorial – 13

1) (-3, -2) 2) $(-\infty, 4] \cup [5, \infty)$ 3) $\left(-\infty, -\frac{7}{2}\right] \cup [0, \infty)$

4) $(-\infty, -2] \cup [2, \infty)$ 5) [-6, 3] 6) $(-\infty, -3) \cup (6, \infty)$

7)	$(-\infty, \infty)$ or $\mathbb{R}$	8)	(-1, 2)	9)	(-1, 4)	10)	[4, 9]

Tutorial – 14

2)	A(3, 3)	B(7, 2)	C(4, 0)	D(0, 3)
	E(-2, 5)	F(-5, 2)	G(-2, 0)	H(-4, -2)
	I(-2, -4)	J(0, -4)	K(2, -3)	L(6, -3)

Question IV : (8)

1)	I	2)	II	3)	IV	4)	III
5)	x-axis	6)	y-axis	7)	x-axis	8)	y-axis

Tutorial – 15

1) 5 *units*
2) $\sqrt{160}$ *units*
3) $\sqrt{26}$ *units*
4) $\frac{\sqrt{37}}{2}$ *units*
5) $\sqrt{x^2 + y^2}$ *units*
6) $\left(\frac{7}{2}, 5\right)$
7) (1, -4)
8) $\left(0, \frac{3}{2}\right)$
9) (1, 0)
10) $\left(\frac{x}{2}, \frac{y}{2}\right)$
11) A(5, 4) is closer to the origin.
12) 8 *units*
13) 10 *units*
14) $\sqrt{50}$ *units*
15) 5 *seconds*
16) 5 *m/minutes*

Tutorial – 16

1) $-\frac{1}{2}$
2) 3
3) $-\frac{4}{5}$
4) $\frac{1}{3}$
5) -1
6) $-\frac{4}{3}$
7) $-\frac{9}{2}$
8) 2
9) Slope of $l_1 = 3$ and Slope of $l_2 = -1$
10) $y = 3x - 1$
11) $2x + y - 3 = 0$
12) $2y = 5x - 1$
13) $x + 3y - 4 = 0$
14) $2x - 3y - 12 = 0$
15) $2y = -3x + 2$
16) $m = -1, b = 1$
17) $m = \frac{3}{2}, b = -3$
18) $m = 4, b = 0$
19) $m = -\frac{3}{2}, b = -\frac{1}{2}$
20) $m = \frac{2}{3}, b = 4$
21) $m = \frac{3}{4}, b = -3$
22) $m = 0, b = \frac{3}{2}$

Tutorial – 17

1)	*Not parallel*	2)	*Parallel*	3)	*Perpendicular*
4)	*Not perpendicular*	5)	$2x - y - 2 = 0$	6)	$3x - 2y + 6 = 0$
7)	$x + 2y - 6 = 0$	8)	$2x + 3y + 6 = 0$		

12 *i)* *The given points are collinear* 12*ii)* *The given points are not collinear.*

Tutorial – 18

1. (a) $f(40) = 13$, because the input 40 (in thousands of people) gives the output 13 (in tons of garbage)
   (b) $f(5) = 2$, means that 5000 people produce 2 tons of garbage per week.

3. (a) In 1995 (5 years after 1990) there were 30 ducks in the lake.
   (b) In 2000 (10 years after 1990) there were 40 ducks in the lake.

5. Graphs (a) (b) (d) and (e) represent *y* as a function of *x* because for every value of *x* there is only one value for *y*. Graphs (c) and (f) are not functions because they contain points that have more than one output for a given input, or values for *x* that have 2 or more values for *y*.

7. Tables (a) and (b) represent *y* as a function of *x* because for every value of *x* there is only one value for *y*. Table (c) is not a function because for the input x=10, there are two different outputs for *y*.

9. Tables (a) (b) and (d) represent *y* as a function of *x* because for every value of *x* there is only one value for *y*. Table (c) is not a function because for the input *x*=3, there are two different outputs for *y*.

11. (a) $f(1) = 1$ (b) $f(3) = 1$

13. (a) $g(2) = 4$ (b) $g(-3) = 2$

15. (a) $f(3) = 53$ (b) $f(2) = 1$

17. $f(-2) = 4 - 2(-2) = 4 + 4 = 8, f(-1) = 6, f(0) = 4, f(1) = 4 - 2(1) = 4 - 2 = 2, f(2) = 0$

19. $f(-2) = 8(-2)^2 - 7(-2) + 3 = 8(4) + 14 + 3 = 32 + 14 + 3 = 49, f(-1) = 18, f(0) = 3, f(1) = 8(1)^2 - 7(1) + 3 = 8 - 7 + 3 = 4, f(2) = 21$

21. $f(-2) = -(-2)^3 + 2(-2) = -(-8) - 4 = 8 - 4 = 4, f(-1) = -(-1)^3 + 2(-1) = -(-1) - 2 = -1, f(0) = 0, f(1) = -(1)^3 + 2(1) = 1, f(2) = -4$

23. $f(-2) = 3 + \sqrt{(-2) + 3} = 3 + \sqrt{1} = 3 + 1 = 4, f(-1) = \sqrt{2} + 3 \approx 4.41, f(0) = \sqrt{3} + 3 \approx 4.73, f(1) = 3 + \sqrt{(1) + 3} = 3 + \sqrt{4} = 3 + 2 = 5, f(2) = \sqrt{5} + 3 \approx 5.23$

25. $f(-2) = ((-2)-2)((-2)+3) = (-4)(1) = -4, f(-1) = -6, f(0) = -6, f(1) = ((1)-2)((1)+3) = (-1)(4) = -4, f(2) = 0$

27. $f(-2) = \frac{(-2)-3}{(-2)+1} = \frac{-5}{-1} = 5, f(-1) = undefined, f(0) = -3, f(1) = -1, f(2) = -1/3$

29. $f(-2) = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}, f(-1) = \frac{1}{2}, f(0) = 1, f(1) = 2, f(2) = 4$

31. Using $f(x) = x^2 + 8x - 4$: $f(-1) = (-1)^2 + 8(-1) - 4 = 1 - 8 - 4 = -11$; $f(1) = 1^2 + 8(1) - 4 = 1 + 8 - 4 = 5$.

(a) $f(-1) + f(1) = -11 + 5 = -6$

(b) $f(-1) - f(1) = -11 - 5 = -16$

33. Using $f(t) = 3t + 5$:

(a) $f(0) = 3(0) + 5 = 5$

(b) $3t + 5 = 0$

$t = -\frac{5}{3}$

35. (a) $y = x$ (iii. Linear)

(b) $y = x^3$ (viii. Cubic)

(c) $y = \sqrt[3]{x}$ (i. Cube Root)

(d) $y = \frac{1}{x}$ (ii. Reciprocal)

(e) $y = x^2$ (vi. Quadratic)

(f) $y = \sqrt{x}$ (iv. Square Root)

(g) $y = |x|$ (v. Absolute Value)

(h) $y = \frac{1}{x^2}$ (vii. Reciprocal Squared)

Tutorial – 19

1. The domain is $[-5, 3)$; the range is $[0, 2]$

3. The domain is $2 < x \leq 8$; the range is $6 \leq y < 8$

5. Since the function is not defined when there is a negative number under the square root, x cannot be less than 2 (it can be equal to 2, because $\sqrt{0}$ is defined). So the domain is $x \geq 2$. Because the inputs are limited to all numbers greater than 2, the number under the square root will always be positive, so the outputs will be limited to positive numbers. So the range is $f(x) \geq 0$.

7. Since the function is not defined when there is a negative number under the square root, x cannot be greater than 3 (it can be equal to 3, because $\sqrt{0}$ is defined). So the domain is $x \leq 3$. Because the inputs are limited to all numbers less than 3, the number under the square root will always be positive, and there is no way for 3 minus a positive number to equal more than three, so the outputs can be any number less than 3. So the range is $f(x) \leq 3$.

9. Since the function is not defined when there is division by zero, x cannot equal 6. So the domain is all real numbers except 6, or $\{x|x \in \mathbb{R}, x \neq 6\}$. The outputs are not limited, so the range is all real numbers, or $\{y \in \mathbb{R}\}$.

11. Since the function is not defined when there is division by zero, x cannot equal $-1/2$. So the domain is all real numbers except $-1/2$, or $\{x|x \in \mathbb{R}, x \neq -1/2\}$. The outputs are not limited, so the range is all real numbers, or $\{y \in \mathbb{R}\}$.

13. Since the function is not defined when there is a negative number under the square root, x cannot be less than -4 (it can be equal to -4, because $\sqrt{0}$ is defined). Since the function is also not defined when there is division by zero, x also cannot equal 4. So the domain is all real numbers less than -4 excluding 4, or $\{x|x \geq -4, x \neq 4\}$. There are no limitations for the outputs, so the range is all real numbers, or $\{y \in \mathbb{R}\}$.

15. It is easier to see where this function is undefined after factoring the denominator. This gives $f(x) = \frac{x-3}{(x+11)(x-2)}$. It then becomes clear that the denominator is undefined when $x = -11$ and when $x = 2$ because they cause division by zero. Therefore, the domain is $\{x|x \in \mathbb{R}, x \neq -11, x \neq 2\}$. There are no restrictions on the outputs, so the range is all real numbers, or $\{y \in \mathbb{R}\}$.

17. $f(-1) = -4; f(0) = 6; f(2) = 20; f(4) = 24$

19. $f(-1) = -1; f(0) = -2; f(2) = 7; f(4) = 5$

21. $f(-1) = -5; f(0) = 3; f(2) = 3; f(4) = 16$

Tutorial – 20

1. $f(g(0)) = 4(7) + 8 = 26,\ g(f(0)) = 7 - (8)^2 = -57$

3. $f(g(0)) = \sqrt{(12) + 4} = 4,\ g(f(0)) = 12 - (2)^3 = 4$

5. $f(g(8)) = 4$

7. $g(f(5)) = 9$

9. $f(f(4)) = 4$

11. $g(g(2)) = 7$

13. $f(g(3)) = 0$

15. $g(f(1)) = 4$

17. $f(f(5)) = 3$

19. $g(g(2)) = 2$

21. $f(g(x)) = \frac{1}{\left(\frac{7}{x}+6\right)-6} = \frac{x}{7}$, $g\left(f(x)\right) = \frac{7}{\left(\frac{1}{x-6}\right)} + 6 = 7x - 36$

23. $f(g(x)) = (\sqrt{x+2})^2 + 1 = x + 3, g(f(x)) = \sqrt{(x^2+1)+2} = \sqrt{(x^2+3)}$

25. $f(g(x)) = |5x + 1|$, $g(f(x)) = 5|x| + 1$

Tutorial – 21

1. $(180°)\left(\frac{\pi}{180°}\right) = \pi$

3. $\left(\frac{5\pi}{6}\right)\left(\frac{180°}{\pi}\right) = 150°$

5. $r = 7 \text{ m}, \theta = 5 \text{ rad}, s = \theta r \rightarrow s = (7 \text{ m})(5) = 35 \text{ m}$

7. $r = 12 \text{ cm}, \theta = 120° = \frac{2\pi}{3}, s = \theta r \rightarrow s = (12 \text{ cm})(\frac{2\pi}{3}) = 8\pi\text{cm}$

9. $r = 6 \text{ ft}, s = 3 \text{ ft}, s = r\theta \rightarrow \theta = s/r$

   Plugging in we have $\theta = (3 \text{ ft})/(6 \text{ ft}) = 1/2$ rad.

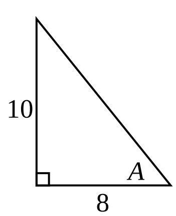
   $(1/2)((180°)/(\pi)) = 90/\pi° \approx 28.6479°$

11. $\theta = 45° = \frac{\pi}{4}$ radians, $r = 6$ cm

    $A = \frac{1}{2}\theta r^2$; we have $A = \frac{1}{2}\left(\frac{\pi}{4}\right)(6 \text{ cm})^2 = \frac{9\pi}{2} \text{ cm}^2$

Tutorial – 22

1.



hypotenuse^2 = $10^2 + 8^2 = 164$ => hypotenuse = $\sqrt{164} = 2\sqrt{41}$

Therefore, sin (A) = $\frac{\text{opposite}}{\text{hypotenuse}} = \frac{10}{2\sqrt{41}} = \frac{5}{\sqrt{41}}$

cos (A) = $\frac{\text{adjacent}}{\text{hypotenuse}} = \frac{8}{2\sqrt{41}} = \frac{4}{\sqrt{41}}$

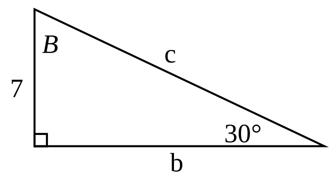
tan (A) = $\frac{\sin(A)}{\cos(A)} = \frac{\frac{5}{\sqrt{41}}}{\frac{4}{\sqrt{41}}} = \frac{5}{4}$ or tan (A) = $\frac{\text{opposite}}{\text{adjacent}} = \frac{10}{8} = \frac{5}{4}$

$$\sec(A) = \frac{1}{\cos(A)} = \frac{1}{\frac{4}{\sqrt{41}}} = \frac{\sqrt{41}}{4}$$

$$\csc(A) = \frac{1}{\sin(A)} = \frac{1}{\frac{5}{\sqrt{41}}} = \frac{\sqrt{41}}{5}$$

and $\cot(A) = \frac{1}{\tan(A)} = \frac{1}{\frac{5}{4}} = \frac{4}{5}$.

3.



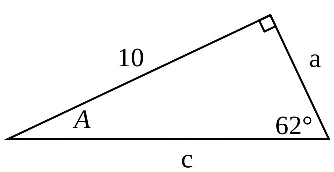
$\sin(30°) = \frac{7}{c} \Rightarrow c = \frac{7}{\sin(30°)} = \frac{7}{\frac{1}{2}} = 14$

$\tan(30°) = \frac{7}{b} \Rightarrow b = \frac{7}{\tan(30°)} = \frac{7}{\frac{1}{\sqrt{3}}} = 7\sqrt{3}$

or $7^2 + b^2 = c^2 = 14^2 \Rightarrow b^2 = 14^2 - 7^2 = 147 \Rightarrow b = \sqrt{147} = 7\sqrt{3}$

$\sin(B) = \frac{b}{c} = \frac{7\sqrt{3}}{14} = \frac{\sqrt{3}}{2} \Rightarrow B = 60°$ or $B = 90° - 30° = 60°$.

5.

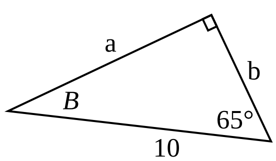


$\sin(62°) = \frac{10}{c} \Rightarrow c = \frac{10}{\sin(62°)} \approx 11.3257$

$\tan(62°) = \frac{10}{a} \Rightarrow a = \frac{10}{\tan(62°)} \approx 5.3171$

$A = 90° - 62° = 28°$

7.

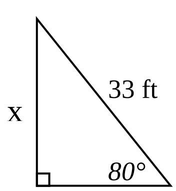


$B = 90° - 65° = 25°$

$\sin(B) = \sin(25°) = \frac{b}{10} \Rightarrow b = 10\sin(25°) \approx 4.2262$

$\cos(B) = \cos(25°) = \frac{a}{10} \Rightarrow a = 10\cos(25°) \approx 9.0631$

9.

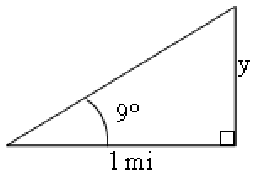


Let x (feet) be the height that the ladder reaches up.

Since $\sin(80°) = \frac{x}{33}$

So the ladder reaches up to $x = 33\sin(80°) \approx 32.4987$ ft of the building.

11.

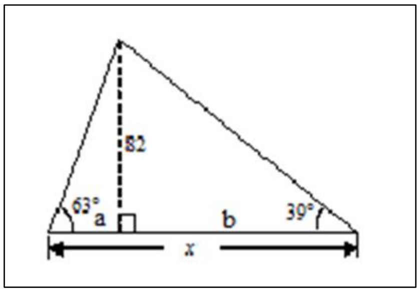


Let y (miles) be the height of the building. Since tan (9°) = $\frac{y}{1}$ = y, the height of the building is y = tan (9°) mi ≈ 836.26984 ft.

13. We have tan (63°) = $\frac{82}{a}$ => $a = \frac{82}{\tan(63°)}$

tan (39°) = $\frac{82}{b}$ => $b = \frac{82}{\tan(39°)}$

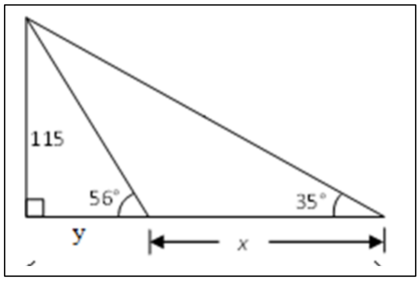
Therefore $x = a + b = \frac{82}{\tan(63°)} + \frac{82}{\tan(39°)}$ ≈ 143.04265.



15. We have tan (35°) = $\frac{115}{z}$ => $z = \frac{115}{\tan(35°)}$

tan (56°) = $\frac{115}{y}$ => $y = \frac{115}{\tan(56°)}$

Therefore $x = z - y = \frac{115}{\tan(35°)} - \frac{115}{\tan(56°)}$ ≈ 86.6685.



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